Mathematical Analysis of a Population Model with Several Competitors On A Single Resource

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1. INTRODUCTION

The purpose of this paper is to present a generalization of Liapunov's Direct Method (4) and use it to prove a result. The result is the competitive exclusion exhibited by the Lotka-Volterra competition equations.

Competitive exclusion is the extinction of some of the species competing with each other for a limited set of resources necessary to their survival. The Principle of Competitive Exclusion refers to a belief in theoretical ecology that only as many species can coexist as there are limiting resources. It states that only one species can best compete for a niche and all species that cannot find a niche for which they compete best will eventually become extinct.

The Lotka-Volterra competition equations are a class of mathematical models presented by Lotka (5) and Volterra (10) and generalized by MacArthur and Levins (6), May (9), and McGehee and Armstrong (8). They depict an ecological system composed of x_1 through x_n species competing for z_1 through z_m resources. The model is described by the differential equations:

$$\dot{x}_1 = x_1 g_1(z_1, \ldots, z_m)$$

$$\cdots$$

$$\dot{x}_n = x_n g_n(z_1, \ldots, z_m)$$

where the g_j's are strictly increasing functions of the z_k's . Throughout this paper "$\dot{x}_j$" means "$\frac{dx_j}{dt}$" . The total amount of each resource is constant and is a linear function of the species plus z_j , the quantity of the j-th resource that is available for comsumption. Volterra considered two species competiting for one resource where g_1 and g_2 are linear functions of z . He argued that if g_1 and g_2 are different one species always became extinct.

Armstrong and McGehee (1) showed that when n species are competing for one resource all but one species become extinct. They divided the positive orthant into two regions and constructed two Liapunov functions. One Liapunov function was used to show that points in the first region were attracted into the second region. The other function showed that all points in the second region eventually converged to one point on an axis of the orthant. This proof is an "ad hoc" argument. It applies only to n species on one resource. There is nothing fundamental in this proof that applies to other systems.

In this paper an alternative proof is presented that seems to be fundamental. It uses a generalization of the Liapunov Direct Method (4). One Liapunov-like function is produced whose properties are used to locate the attracting set. The Liapunov generalization is:

Theorem 1. Let X be a smooth compact manifold where $\varphi(x,t)$ is the flow of the system $\dot{x} = F(x)$ on the manifold, $\varphi \in C^{\infty}(X \times \mathbb{R}, X)$, $V \in C^{\infty}(X, \mathbb{R})$, $\mathcal{M} = \dot{V}^{-1}(0)$, $\mathcal{M}_2 = \ddot{V}^{-1}(0) \cap \mathcal{M}$, $\dddot{V}^{-1}(\mathcal{M}-\mathcal{M}_2) > 0$ and each point in $\mathcal{M}_2$ is isolated in $\mathcal{M}$. If $\mathcal{R}$ is the set of rest points in $\mathcal{M}$ and $x \in X$, then $\omega(x) \subset \mathcal{R}$.

Here, as elsewhere, "$\ddot{V}$" means "$\frac{d^2V}{dt^2}$", $V(\mathcal{M}) > 0$ means that for each $x \in \mathcal{M}$ $V(x) > 0$, $g'(z)$ means $\frac{dg}{dz}$, and $\omega(x)$ is the omega limit set of x and is defined as $\omega(x) = \mathrm{cl}\{\bigcap_{t>0} \varphi(x,[t,\infty))\}$ where cl is set closure.

Theorem 1 will be proved in the last chapter. Notice that for a Liapunov function V, $\mathcal{M} = \mathcal{M}_2 = 0$, the origin, and $\dot{V}(X-\mathcal{M}) < 0$. When these conditions are satisfied $\mathcal{M}_2 = \mathcal{R}$ is trivially isolated in $\mathcal{M}$. V trivially satisfies $\ddot{V}(\mathcal{M}-\mathcal{M}_2) > 0$ since this statement cannot be shown to be false. By Theorem 1, $\omega(x) = 0$. Therefore, Theorem 1 is a generalization of the Liapunov Direct Method.

In the next chapter the theorem is applied to show that there is only one survivor when n species are competing for one resource in the Lotka-Volterra manner. Finally, in the last chapter a lemma is stated and proved and then used to prove the main theorem.

2. BIOLOGICAL APPLICATION

This chapter begins with a description of a Lotka-Volterra competition model. Then the main theorem is presented and used to show competitive exclusion.

X is the subset of the positive orthant of $\mathbb{R}^n$ where z, the amount of available resource, is nonnegative. For each j in the set $\{1,\ldots,n\}$ k_j is a positive real number, $g_j(z)$ is a strictly increasing function of z where $0 \leq z \leq z_0$, z_0 is a positive real number and $\bar{z}_j = g_j^{-1}(0)$ is a positive real number. $\omega(x)$ is the omega limit set (2), the set that x converges to, $\omega(x) = \mathrm{cl}\{\bigcap_{t>0} \varphi(x,[t,\infty))\}$, where cl is set closure and φ is the flow of the system.

The competition model is described by the differential equations and the resource constraint:

$$\dot{x}_1 = x_1 g_1(z)$$

$$- - - - -$$

$$\dot{x}_n = x_n g_n(z)$$

$$z = z_0 - \sum_{j=1}^{n} k_j x_j$$

Corollary 1 (Application): If x lies on X and $\bar{z}_j$ the zeros of $g_j(z)$ have the property that $0 < \bar{z}_1 < \bar{z}_2 < \ldots \leq \bar{z}_n < z_0$ then $\omega(x) \subset \{0,(x_1,0,\ldots,0),\ldots,(0,\ldots,0,x_n) : x_j = \frac{z_0 - \bar{z}_j}{k_j}\}$.

The ordering of the $\{\bar{z}_j\}$ is a comparison of the effectiveness of the species in competing for the resource. The g_j's are growth functions and their zeros indicate the lower bounds of resource at which the jth species can still increase its population. k_j is the amount of resource that each member of species j requires.

Proof of Corollary 1:

Notice that the convex set X is postively invariant in the system on $\mathbb{R}^n$. First we will find $\mathcal{M}$, $\mathcal{M}_2$ and $\mathcal{R}$. Let $V = -z$. $\mathcal{M} = \dot{V}^{-1}(0)$.

$$\dot{V} = -\dot{z} = \sum_{j=1}^{n} k_j \dot{x}_j = \sum_{j=1}^{n} k_j x_j g_j(z) \quad .$$

Therefore, $\mathcal{M} = \{x \in X : \dot{z} = 0\}$

$$\mathcal{M}_2 = \{x \in \mathcal{M} : \ddot{V}(x) = 0\}$$

$$\begin{aligned} \ddot{V} &= \sum_{j=1}^{n} k_j \dot{x}_j g_j(z) + \sum_{j=1}^{n} k_j x_j g_j'(z)\dot{z} \\ &= \sum_{j=1}^{n} k_j x_j g_j^2(z) + \sum_{j=1}^{n} k_j x_j g_j'(z)\dot{z} \quad . \end{aligned}$$

On $\mathcal{M}$, $\dot{z} = 0$. Therefore,

$$\ddot{V}\big|_{\mathcal{M}} = \sum_{j=1}^{n} k_j x_j g_j^2(z) \quad .$$

For each j , $x_j \geq 0$ or $g_j^2(z) \geq 0$.

So on $\mathcal{M}_2$ each term of $\ddot{V}$ must be zero. Therefore $\mathcal{M}_2 = \{x \in \mathcal{M} :$ for each j , $x_j = 0$ or $g_j(z) = 0\}$. It is clear that $0 = (0,\ldots,0) \in \mathcal{M}_2$ and for each $j = 1,\ldots,n$

$\bar{x}_j = (0,\ldots,0,x_j,0,\ldots,0) \in \mathcal{M}_2$. So $\mathcal{M}_2 = \{0,\bar{x}_j : j = 1,\ldots,n\}$.

Notice that each point in $\mathcal{M}_2$ is isolated in $\mathcal{M}$ and is a rest point. Therefore, $\mathcal{R} = \mathcal{M}_2$. Also $\dot{V}(\mathcal{M}-\mathcal{M}_2) > 0$. Applying Theorem 1 $\omega(x) \in \mathcal{M}_2$ for each $x \in X$.

However, we can locate one point in $\mathcal{R}$ which attracts the whole interior of X .

<u>Theorem 2</u>: If x lies in the interior of X and the zeros of $g_j(z)$ have the property that $0 < \bar{z}_1 < \bar{z}_2 < \ldots < \bar{z}_n$ then $\omega(x) = \bar{x}_1 = (x_1,0,\ldots,0)$ where $x_1 = \dfrac{\bar{z}_0 - z_1}{k_1}$.

Proof of Theorem 2:

We will show that only $\bar{x}_1$ has a domain of attraction that intersects the interior of X . From Corollary 1 we know that if x lies in the interior of X , $\omega(x) \in \mathcal{M}_2$. Then it follows that $\omega(x) = \bar{x}_1$.

"If f is a C' flow on a smooth manifold X , x_0 is a hyperbolic fixed point of f , V is a small neighborhood of x_0 and W^+ is the local positively invariant asymptotic set

$$W^+ = \{x \in V : f(x,t) \subset V \text{ for } t > 0 \text{ and } f(x,t) \to x_0 \text{ as } t \to \infty\} ,$$

then W^+ is an embedded submanifold of V with the embedding as smooth as f". (7) This is the Stable Manifold Theorem (3) and W^+ is called the stable manifold of V . We will show that only $\bar{x}_1$ has a stable manifold that intersects the interior of X .

We recall that $\dot{x} = F(x)$. Then $DF = (\frac{\partial}{\partial x_k}(x_j g_j'(z))), j,k = 1,\ldots,n$

$$= \begin{bmatrix} g_1(z) - k_1x_1g_1'(z) , & -k_2x_1g_1'(z),\ldots,-k_nx_1g_1'(z) \\ -k_1x_2g_2'(z) , & g_2(z)-k_2x_2g_2'(z),\ldots,-k_nx_2g_2'(z) \\ - - - - - & - - - - - \\ -k_1x_ng_n'(z) , & -k_2x_ng_n'(z),\ldots,g_n(z)-k_nx_ng_n'(z) \end{bmatrix} .$$

If $\bar{x}_j = (0,\ldots,0,x_j,0,\ldots,0)$ then

$$DF(\bar{x}_j) = \begin{bmatrix} g_1 , & 0 , & \ldots , & 0 , & \ldots , & 0 , & 0 \\ 0 , & g_2 , & \ldots , & 0 , & \ldots , & 0 , & 0 \\ - & - & - & - & - & - & - \\ -k_1x_jg_j', & -k_2x_jg_j', & \ldots, & -k_jx_jg_j', & \ldots , & -k_{n-1}x_jg_j', & -k_nx_jg_j' \\ - & - & - & - & - & - & - \\ 0 , & 0 , & \ldots , & 0 , & \ldots , & g_{n-1} , & 0 \\ 0 , & 0 , & \ldots , & 0 , & \ldots , & 0 , & g_n \end{bmatrix}$$

because $z = \bar{z}_j$, $x_k = 0$ for $k \neq j$ and $g_j(\bar{z}_j) = 0$.

Now we will compare the local stable manifolds about the members of $\mathfrak{R}$.

Case 1: $\bar{x}_1$

The eigenvalues of $DF(\bar{x}_1)$ are

$$\{-k_1 x_1 g_1'(\bar{z}_1) \ , \ g_2(\bar{z}_1) \ , \ g_3(\bar{z}_1), \ldots, g_n(\bar{z}_1)\} \quad .$$

All of them are negative because

(1) k_1 , x_1 and g_1' are positive and

(2) The $\{g_j\}$ are increasing functions of z . So for $j = 2,\ldots,n$ then $\bar{z}_1 < \bar{z}_j$ and thus $g_j(\bar{z}_1) < g_j(\bar{z}_j) = 0$.

Hence, x_1 is asymptotically stable in a sufficiently small neighborhood U . Since U is open in X , U intersects the interior of X . Thus, there is a point x in the interior of X so that $\omega(x) = \bar{x}_1$.

Case 2: $\bar{x}_j$ for $j = 2,\ldots,n$

Here we will produce the eigenvectors and eigenvalues and show that the space spanned by the eigenvectors with negative eigenvalues lies in the boundary of X .

Notice that for $k \neq j$ the eigenvalue $\lambda = g_k$ and the eigenvector v , where $v = (0,\ldots 0,\underset{\substack{\uparrow \\ k\text{th} \\ \text{position}}}{1},0,\ldots,0,\underset{\substack{\uparrow \\ j\text{th} \\ \text{position}}}{\alpha},0,\ldots 0)^T$. Solving for α we have $DF(\bar{x}_j)v = \lambda v$. So $-k_k x_j g_j' - k_j x_j g_j' \alpha = g_k \alpha$.

Thus, $\alpha = -k_k x_j g_j' / (g_k + k_j x_j g_j')$.

When $k = j$, $\lambda = -k_j x_j g_j'$ and $v = (0,\ldots,0,\underset{\substack{\uparrow \\ k\text{th} \\ \text{position}}}{1},0,\ldots,0)$.

So, the eigenvalues of $DF(\bar{x}_j)$ are

$$\{g_1(\bar{z}_j),\ldots,g_{j-1}(\bar{z}_1),-k_jx_jg_j'(\bar{z}_j),g_{j+1}(\bar{z}_j),\ldots,g_n(\bar{z}_j)\} \quad .$$

All of the eigenvalues are either positive or negative because

$$\text{if } h < j \quad 0 = g_h(\bar{z}_h) < g_h(\bar{z}_j)$$
$$\text{if } h > j \quad 0 = g_h(\bar{z}_h) > g_h(\bar{z}_j) \quad , \text{ and}$$
$$-k_jx_jg_j'(\bar{z}_j) < 0 \quad .$$

If B_j is the set of eigenvectors with negative eigenvalues then

$$\begin{aligned} B_j = \{&(0,\ldots,0,1,0,\ldots,0)^T \;, \\ &(0,\ldots,0,1,-k_{j+1}x_jg_j'/g_{j+1}+k_jx_jg_j'),0,\ldots,0)^T \;, \\ &\ldots \;, \\ &(0,\ldots,0,1,0,\ldots,0,-k_nx_jg_j'/(g_n+k_jx_jg_j'))^T\} \end{aligned}$$

where the 1's are in the j-th position.

If $\langle B_j\rangle$ is the subspace spanned by B_j , then for every $x = (x_1,\ldots,x_n)$ in $\langle B_j\rangle$, $x_1 = 0$. Therefore, $\langle B_j\rangle$ is contained in the boundary of X . Since the stable manifold is tangent to the boundary and unique, and the boundary is invariant, then the stable manifold lies in the boundary. Therefore, there does not exist an x contained in the interior of X so that $\omega(x) = \bar{x}_j$ for $j = 2,\ldots,n$.

Case 3: 0

$$DF(0) = \begin{bmatrix} g_1(z_0) & , & 0 & , & \dots & , & 0 \\ 0 & , & g_2(z_0) & , & \dots & , & 0 \\ - & & - & & - & & - \\ 0 & , & 0 & , & \dots & , & g_n(z_0) \end{bmatrix} .$$

The eigenvalues of $DF(0)$ are $\{g_1(z_0),\dots,g_n(z_0)\}$. They are all positive since $\bar{z}_j < z_0$ and $0 = g_j(\bar{z}_j) < g_j(z_0)$. Therefore, the stable manifold of 0 is 0 and 0 is unstable. Hence the stable manifold of 0 does not intersect the interior of X .

In conclusion, $\bar{x}_1$ is the only member of $\mathcal{R}$ whose local stable manifold intersects the interior of X . Therefore only $\bar{x}_1$ can be an attractor for the interior of X . Because $\omega(x)$ is contained in $\mathcal{R}$, $\omega(x) = (x_1, 0, \dots, 0)$.

Since $z = z_0 - \sum_{j=1}^{n} k_j x_j$ and $z = \bar{z}_j$ then $x_1 = \frac{z_0 - \bar{z}_j}{k_j}$. This completes the proof of Theorem 2.

3. PROOF OF THEOREM

In this chapter, Theorem 1 is proved. First a lemma is presented and proved in which we will show that for smooth flows on smooth compact manifolds and for certain real smooth functions on the manifold all orbits of the flow go to an invariant set in the zeros of the first time derivative of the function. Then we will prove the main theorem and describe that invariant set in terms of the second time derivative of the function.

We will assume the following notational conventions. $V(\mathcal{M}) > 0$ means $V(x) > 0$ for $x \in \mathcal{M}$. $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Lemma 1: Let X be a smooth compact manifold, $\varphi(x,t)$ be the flow of the system $\dot{x} = F(x)$, $\varphi \in C^\infty(X \times \mathbb{R}, X)$, $V \in C^\infty(X, \mathbb{R})$, $\mathcal{M} = \dot{V}^{-1}(0)$ and $\mathcal{N}$ is the maximal invariant subset of $\mathcal{M}$. Assume that $\varphi(\mathcal{M} - \mathcal{N}, [0,\infty))$ does not intersect $\dot{V}^{-1}(\mathbb{R}^-)$. Then $\omega(x) \subset \mathcal{N}$.

Proof:

Notice that $\omega(x)$ exists, and is invariant and connected.

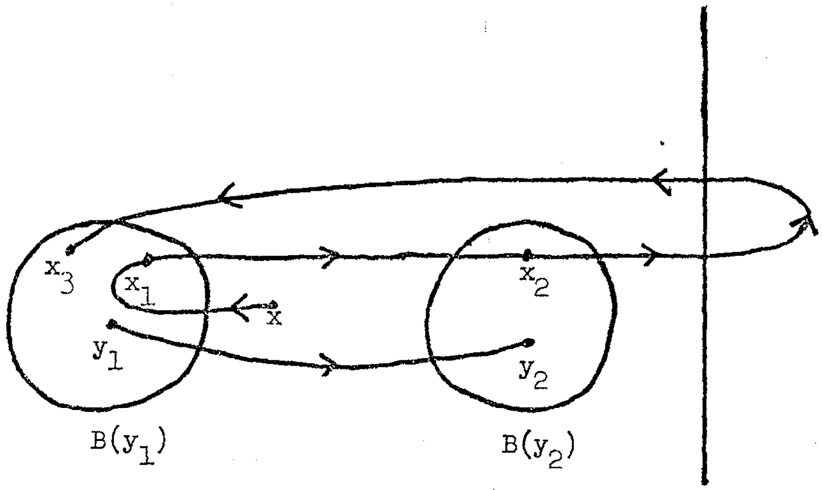

Figure 1: $\omega(x)$ cannot lie in $\dot{V}^{-1}(\mathbb{R}^+)$

We claim that $\omega(x) \cap \dot{V}^{-1}(\mathbb{R}^+) = \emptyset$. Suppose this is not so. Then there exists $y_1 \in \omega(x) \cap \dot{V}^{-1}(\mathbb{R}^+)$ and $t > 0$. So that $y_2 = \varphi(y_1, t) \in \dot{V}^{-1}(\mathbb{R}^+)$ and $V(y_2) > V(y_1)$. We can find $B(y_1)$ and $B(y_2)$, neighborhoods of y_1 and y_2 respectively, where $V(B(y_1)) < V(B(y_2))$. Since $y_1, y_2 \in \omega(x)$ we can find $s_3 > s_2 > s_1 > 0$ so that $x_1 = \varphi(x, s_1) \in B(y_1)$, $x_2 = \varphi(x, s_2) \in B(y_2)$ and $x_3 = \varphi(x, s_3) \in B(y_1)$. Thus $V(x_3) > V(x_2)$. So on $\varphi(x, [s_2, s_3])$ V is somewhere decreasing. Therefore $\varphi(x, [s_2, s_3])$ intersects $\mathcal{M} - \mathcal{N}$ and flows into $\dot{V}^{-1}(\mathbb{R}^{-1})$. This is a contradiction to the hypothesis. Thus $\omega(x) \cap \dot{V}^{-1}(\mathbb{R}^+) = \emptyset$.

We claim that $\omega(x) \cap \dot{V}^{-1}(\mathbb{R}^-) = \emptyset$. Suppose this is not so. Then in a manner similiar to the previous argument we can choose $y_1, y_2 \in \omega(x) \cap \dot{V}^{-1}(\mathbb{R}^-)$ and neighborhoods $B(y_1)$ and $B(y_2)$ so that

$V(B(y_1)) > V(B(y_2))$. Selecting $x_1 \in B(y_1)$ we can follow its orbit from $B(y_1)$ into $B(y_2)$ and back to $B(y_1)$. Therefore, at some point the orbit must be increasing. Thus, it must flow from $\dot{V}^{-1}(\mathbb{R}^-)$ into $\dot{V}^{-1}(\mathbb{R}^+)$ and back to $V(\mathbb{R}^-)$ crossing $\mathcal{M} - \mathcal{N}$ in each direction. Consequently, it orbit must cross from $\dot{V}^{-1}(\mathbb{R}^+)$, through $\mathcal{M} - \mathcal{N}$, and into $\dot{V}^{-1}(\mathbb{R}^-)$. Again, this contradicts the hypothesis. Therefore, $\omega(x) \cap \dot{V}^{-1}(\mathbb{R}^-) = \emptyset$.

Therefore, $\omega(x) \subset \mathcal{M}$.

Since $\omega(x)$ is invariant and $\mathcal{N}$ contains all invariant subsets of $\mathcal{M}$, then $\omega(x) \subset \mathcal{N}$. This completes the proof of Lemma 1.

Theorem 1 (The Main Theorem): Let X be a smooth compact manifold where $\varphi(x,t)$ is the flow of the system $\dot{x} = F(x)$, $\varphi \in C^\infty(X \times \mathbb{R}, X)$, $V \in C^\infty(X,\mathbb{R})$, $\mathcal{M} = \dot{V}^{-1}(0)$, $\mathcal{M}_2 = \ddot{V}^{-1}(0) \cap \mathcal{M}$, $\dddot{V}^{-1}(\mathcal{M}-\mathcal{M}_2) > 0$, and each point in $\mathcal{M}_2$ is isolated in $\mathcal{M}$. If $\mathcal{R}$ is the set of rest points in $\mathcal{M}$ and $x \in X$, then $\omega(x) \subset \mathcal{R}$.

Proof:

To establish the theorem it is sufficient to prove that $\mathcal{R}$ is the maximal invariant subset of $\mathcal{M}$ and that $\varphi(\mathcal{M}-\mathcal{R},[0,\infty)) \cap \dot{V}^{-1}(\mathbb{R}^-) = \emptyset$. Then, the theorem follows from the corollary.

First we will show that $\mathcal{R}$ is the maximal positively invariant subset of $\mathcal{M}$.

Let $y \in \mathcal{M}-\mathcal{R}$. Then y is not a rest point. Since $\mathcal{M}_2$ is isolated, there is a small t so that $\varphi(y,t) \notin \mathcal{M}_2$. If $\varphi(y,t) \notin \mathcal{M}$, then y is not contained in an invariant subset of $\mathcal{M}$.

If $\varphi(y,t) \in \mathcal{M}$, then $\varphi(y,t) \in \mathcal{M}-\mathcal{M}_2$. But $\mathcal{M}-\mathcal{M}_2$ flows into $\dot{V}^{-1}(\mathbb{R}^+)$. Thus, $\mathcal{M}-\mathcal{R}$ does not contain points which lie in invariant subsets of $\mathcal{M}$. On the other hand, $\mathcal{R}$ is invariant. Therefore, $\mathcal{R}$ is the maximal invariant subset of $\mathcal{M}$.

Now we will show that the positive flow from $\mathcal{M}-\mathcal{R}$ and $t > 0$ $\varphi(y,t) \notin \dot{V}^{-1}(\mathbb{R}^-)$. Suppose not.

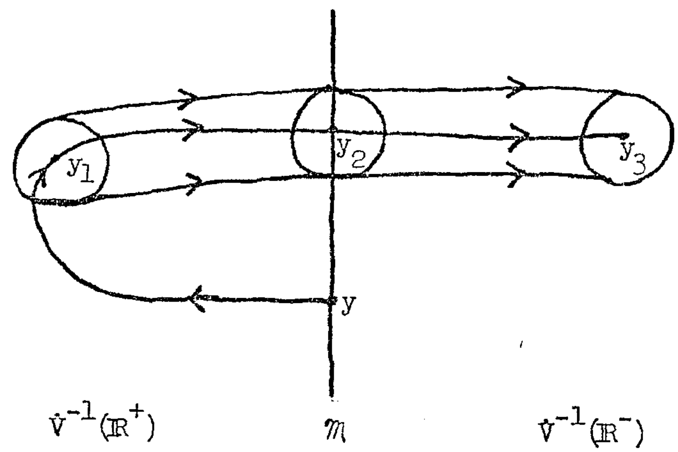

Figure 2: $\mathcal{M}-\mathcal{R}$ avoids $\dot{V}^{-1}(\mathbb{R}^-)$

We have seen that $\varphi(y,[0,\infty))$ initially crosses into $\dot{V}^{-1}(\mathbb{R}^+)$. Because $\mathcal{M}$ separates $\dot{V}^{-1}(\mathbb{R}^+)$ and $\dot{V}^{-1}(\mathbb{R}^-)$, and φ is smooth, there exist $y_1 \in \dot{V}^{-1}(\mathbb{R}^+)$, $y_2 \in \mathcal{M}$, $y_3 \in \dot{V}^{-1}(\mathbb{R}^-)$ and $t_3 > t_2 > t_1 > 0$ so that $y_1 = \varphi(y,t_1)$, $y_2 = \varphi(y,t_2)$, $y_3 = \varphi(y,t_3)$ and $\varphi(y,(t_1,t_2)) \subset \dot{V}^{-1}(\mathbb{R}^+)$, and $\varphi(y,(t_2,t_3)) \subset \dot{V}^{-1}(\mathbb{R}^-)$.

Since $\mathcal{M}_2$ is an isolated set there is a neighborhood $B(y_2)$ that contains no members of $\mathcal{M}_2$ other than y_2. Since

$\varphi(y,[t_2,t_3])$ is compact and V is differentiable we may choose $B(y_1)$ small enough so that for every $z \in B(y_1)$, $\varphi(z,[0,\infty))$ crosses $\mathcal{M}$ in $B(y_2)$. Therefore if z is not in the orbit of y , it must cross $\mathcal{M}$ in $\mathcal{M} - \mathcal{M}_2$. This contradicts the assumption that $\ddot{V}(\mathcal{M} - \mathcal{M}_2) > 0$. Therefore, $\mathcal{M} - \mathcal{M}_2$ avoids $\dot{V}^{-1}(\mathbb{R}^-)$ in the positive flow. This completes the proof.

APPENDIX: RESULTS

Lemma 1: Let X be a smooth compact manifold, $\varphi(x,t)$ be the flow of the system $\dot{x} = F(x)$, $\varphi \in C^{\infty}(X \times \mathbb{R}, X)$, $V \in C^{\infty}(X, \mathbb{R})$, $\mathcal{M} = \dot{V}^{-1}(0)$ and $\mathcal{N}$ is the maximal invariant subset of $\mathcal{M}$. Assume that $\varphi(\mathcal{M} - \mathcal{N}, [0,\infty))$ does not intersect $\dot{V}^{-1}(\mathbb{R}^{-})$. Then $\omega(x) \subset \mathcal{N}$.

Theorem 1: Let X be a smooth compact manifold where $\varphi(x,t)$ is the flow of the system $\dot{x} = F(x)$, $\varphi \in C^{\infty}(X \times \mathbb{R}, X)$, $V \in C^{\infty}(X, \mathbb{R})$, $\mathcal{M} = \dot{V}^{-1}(0)$, $\mathcal{M}_2 = \ddot{V}^{-1}(0) \cap \mathcal{M}$, $\ddot{V}^{-1}(\mathcal{M} - \mathcal{M}_2) > 0$, and each point in $\mathcal{M}_2$ is isolated in $\mathcal{M}$. If $\mathcal{R}$ is the set of rest points in $\mathcal{M}$ and $x \in X$, then $\omega(x) \in \mathcal{R}$.

Corollary 1: If X is the subset of the positive orthant of $\mathbb{R}^n$ where $\dot{z}$ is nonnegative with the system:

$$\dot{x}_1 = x_1 g_1(z)$$

$$\cdots$$

$$\dot{x}_n = x_n g_n(z) \qquad ,$$

$$z = z_0 - \sum_{j=1}^{n} k_j x_j$$

where $k_j > 0$, $g_j^1 > 0$, $\bar{z}_j = g_j^{-1}(0)$, $0 < \bar{z}_1 < \ldots < \bar{z}_n < z_0$, $\bar{x}_j = (0,\ldots,0,x_j,0,\ldots,0)$, and $x_j = \dfrac{z_0 - \bar{z}_j}{k_j}$. If $x \in X$, then $\omega(x) \subset \{0, \bar{x}_1, \ldots, \bar{x}_n\}$.

Theorem 2: With the same hypothesis as Corollary 1, if x lies in the interior of X , then $\omega(x) = \bar{x}_1$.

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