Recovery of Low-Rank Plus Compressed Sparse Matrices
with Application to Unveiling Traffic Anomalies†

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Abstract

Given the superposition of a low-rank matrix plus the product of a known fat compression matrix times a sparse matrix, the goal of this paper is to establish deterministic conditions under which exact recovery of the low-rank and sparse components becomes possible. This fundamental identifiability issue arises with traffic anomaly detection in backbone networks, and subsumes compressed sensing as well as the timely low-rank plus sparse matrix recovery tasks encountered in matrix decomposition problems. Leveraging the ability of $\ell_1$- and nuclear norms to recover sparse and low-rank matrices, a convex program is formulated to estimate the unknowns. Analysis and simulations confirm that the said convex program can recover the unknowns for sufficiently low-rank and sparse enough components, along with a compression matrix possessing an isometry property when restricted to operate on sparse vectors. When the low-rank, sparse, and compression matrices are drawn from certain random ensembles, it is established that exact recovery is possible with high probability. First-order algorithms are developed to solve the nonsmooth convex optimization problem with provable iteration complexity guarantees. Insightful tests with synthetic and real network data corroborate the effectiveness of the novel approach in unveiling traffic anomalies across flows and time, and its ability to outperform existing alternatives.

Index Terms

Sparsity, low rank, convex optimization, identifiability, traffic volume anomalies.

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I. INTRODUCTION

Let $X_0 \in \mathbb{R}^{L \times T}$ be a low-rank matrix [$r := \text{rank}(X_0) \ll \min(L, T)$], and let $A_0 \in \mathbb{R}^{F \times T}$ be sparse ($s := \|A_0\|_0 \ll FT$, $\|\cdot\|_0$ counts the nonzero entries of its matrix argument). Given a compression matrix $R \in \mathbb{R}^{L \times F}$ with $L \leq F$, and observations

$$Y = X_0 + RA_0$$

the present paper deals with the recovery of $\{X_0, A_0\}$. This task is of interest e.g., to unveil anomalous flows in backbone networks [23], [25], [39], to extract the time-varying foreground from a sequence of compressed video frames [37], or, to identify active brain regions from undersampled functional magnetic resonance imagery (fMRI) [15]. In addition, this fundamental problem is found at the crossroads of compressive sampling (CS), and the timely low-rank-plus-sparse matrix decompositions.

In the absence of the low-rank component ($X_0 = 0_{L \times T}$), one is left with an under-determined sparse signal recovery problem; see e.g., [12], [31] and the tutorial account [13]. When $Y = X_0 + A_0$, the formulation boils down to principal components pursuit (PCP), also referred to as robust principal component analysis (PCA) [10], [14], [18]. For this idealized noise-free setting, sufficient conditions for exact recovery are available for both of the aforementioned special cases. However, the superposition of a low-rank and a compressed sparse matrix in (1) further challenges identifiability of $\{X_0, A_0\}$. In the presence of ‘dense’ noise, stable reconstruction of the low-rank and sparse matrix components is possible via PCP [38], [40]. Earlier efforts dealing with the recovery of sparse vectors in noise led to similar performance guarantees; see e.g., [5] and references therein. Even when $X_0$ is nonzero, one could envision a CS variant where the measurements are corrupted with correlated (low-rank) noise [15]. Last but not least, when $A_0 = 0_{F \times T}$ and $Y$ is noisy, the recovery of $X_0$ subject to a rank constraint is nothing else than PCA – arguably, the workhorse of high-dimensional data analysis [22].

The main contribution of this paper is to establish that given $Y$ and $R$ in (1), for small enough $r$ and $s$ one can exactly recover $\{X_0, A_0\}$ by solving the nonsmooth convex optimization problem

$$\min_{\{X, A\}} \|X\|_* + \lambda \|A\|_1$$

subject to $Y = X + RA$

where $\lambda \geq 0$ is a tuning parameter; $\|X\|_* := \sum \sigma_i(X)$ is the nuclear norm of $X$ ($\sigma_i$ stands for the $i$-th singular value); and, $\|X\|_1 := \sum_{i,j} |x_{ij}|$ denotes the $\ell_1$-norm. The aforementioned norms are convex surrogates to the rank and $\ell_0$-norm, respectively, which albeit natural as criteria they are NP-hard to optimize [16], [28]. Recently, a greedy algorithm for recovering low-rank and sparse matrices from
compressive measurements was put forth in [37]. However, convergence of the algorithm and its error performance are only assessed via numerical simulations. A recursive algorithm capable of processing data in real time can be found in [15], which attains good performance in practice but does not offer theoretical guarantees.

A deterministic approach along the lines of [14] is adopted first to derive conditions under which (1) is locally identifiable (Section II). Introducing a notion of incoherence between the additive components $X_0$ and $RA_0$, and resorting to the restricted isometry constants of $R$ [12], sufficient conditions are obtained to ensure that (P1) succeeds in exactly recovering the unknowns (Section III-A). Intuitively, the results here assert that if $r$ and $s$ are sufficiently small, the nonzero entries of $A_0$ are sufficiently spread out, and subsets of columns of $R$ behave as isometries, then (P1) exactly recovers $\{X_0, A_0\}$. As a byproduct, recovery results for PCP and CS are also obtained by specializing the aforesaid conditions accordingly (Section III-B). The proof of the main result builds on Lagrangian duality theory [3], [8], to first derive conditions under which $\{X_0, A_0\}$ is the unique optimal solution of (P1) (Section IV-A). In a nutshell, satisfaction of the optimality conditions is tantamount to the existence of a valid dual certificate. Stemming from the unique challenges introduced by $R$, the dual certificate construction procedure of Section IV-B is markedly distinct from the direct sum approach in [14], and the (random) golfing scheme of [10]. Section V shows that low-rank, sparse, and compression matrices drawn from certain random ensembles satisfy the sufficient conditions for exact recovery with high probability.

Two iterative algorithms for solving (P1) are developed in Section VI, which are based on the accelerated proximal grandient (APG) method [2], [24], [29], [30], and the alternating-direction method of multipliers (AD-MoM) [4], [8]. Numerical tests corroborate the exact recovery claims, and the effectiveness of (P1) in unveiling traffic volume anomalies from real network data (Section VII). Section VIII concludes the paper with a summary and a discussion of limitations, possible extensions, and interesting future directions. Technical details are deferred to the Appendix.

A. Notational conventions

Bold uppercase (lowercase) letters will denote matrices (column vectors), and calligraphic letters will denote sets. Operators $(\cdot)'$, $(\cdot)\dagger$, $\text{tr}(\cdot)$, $\text{vec}(\cdot)$, $\text{diag}(\cdot)$, $\lambda_{\text{max}}(\cdot)$, $\sigma_{\text{min}}(\cdot)$, and $\otimes$ will denote transposition, matrix pseudo inverse, matrix trace, matrix vectorization, diagonal matrix, spectral radius, minimum singular value, and Kronecker product, respectively; $|\cdot|$ will be used for the cardinality of a set and the magnitude of a scalar. The $n \times n$ identity matrix will be represented by $I_n$ and its $i$-th column by $e_i$; while $0_n$ denotes the $n \times 1$ vector of all zeros, and $0_{n \times p} := 0_n 0_p'$. The $\ell_q$-norm of vector $x \in \mathbb{R}^p$ is
Lemma 1: Matrix $Y$ uniquely decomposes into $X_0 + RA_0$ if and only if $\Phi \cap \Omega_R = \{0_{L \times T}\}$, and
\( \mathbf{RH} \neq \mathbf{0}_{L \times T}, \forall \mathbf{H} \in \Omega \setminus \{\mathbf{0}_{F \times T}\}. \)

**Proof:** Since by definition \( \mathbf{X}_0 \in \Phi \) and \( \mathbf{A}_0 \in \Omega \), one can represent every element in the subspaces \( \Phi \) and \( \Omega_R \) as \( \mathbf{X}_0 + \mathbf{Z}_1 \) and \( \mathbf{RA}_0 + \mathbf{Z}_2 \), respectively, where \( \mathbf{Z}_1 \in \Phi \) and \( \mathbf{Z}_2 \in \Omega_R \). Assume that \( \Phi \cap \Omega_R = \{0_{L \times T}\} \), and suppose by contradiction that there exist nonzero perturbations \( \{\mathbf{Z}_1, \mathbf{Z}_2\} \) such that
\[
\mathbf{Y} = \mathbf{X}_0 + \mathbf{Z}_1 + \mathbf{RA}_0 + \mathbf{Z}_2.
\]
Then, \( \mathbf{Z}_1 + \mathbf{Z}_2 = 0_{L \times T} \), meaning that \( \mathbf{Z}_1 \) and \( \mathbf{Z}_2 \) belong to the same subspace, which contradicts the assumption. Conversely, suppose there exists a non-zero \( \mathbf{Z} \in \Omega_R \cap \Phi \). Clearly, \( \{\mathbf{X}_0 + \mathbf{Z}, \mathbf{RA}_0 - \mathbf{Z}\} \) is a feasible solution where \( \mathbf{X}_0 + \mathbf{Z} \in \Phi \) and \( \mathbf{RA}_0 - \mathbf{Z} \in \Omega_R \). This contradicts the uniqueness assumption. In addition, the condition \( \mathbf{RH} \neq 0, \mathbf{H} \in \Omega \setminus \{0_{L \times T}\} \) ensures that \( \mathbf{Z} = 0_{L \times T} \in \Phi \cap \Omega_R \) only when \( \mathbf{Z} = \mathbf{RH} = 0_{L \times T} \) for \( \mathbf{H} = 0_{F \times T} \).

In words, (1) is locally identifiable if and only if the subspaces \( \Phi \) and \( \Omega_R \) intersect transversally, and the sparse matrices in \( \Omega \) are not annihilated by \( \mathbf{R} \). This last condition is unique to the setting here, and is not present in [10] or [14].

**Remark 1 (Projection operators):** Operator \( \mathcal{P}_\Omega(\mathbf{X}) \ (\mathcal{P}_\Omega^\perp(\mathbf{X})) \) denotes the orthogonal projection of \( \mathbf{X} \) onto the subspace \( \Omega \) (orthogonal complement \( \Omega^\perp \)). It simply sets those elements of \( \mathbf{X} \) not in \( \text{supp}(\mathbf{A}_0) \) to zero. Likewise, \( \mathcal{P}_\Phi(\mathbf{X}) \ (\mathcal{P}_{\Phi^\perp}(\mathbf{X})) \) denotes the orthogonal projection of \( \mathbf{X} \) onto the subspace \( \Phi \) (orthogonal complement \( \Phi^\perp \)). Let \( \mathbf{P}_U := \mathbf{UU}' \) and \( \mathbf{P}_V := \mathbf{VV}' \) denote, respectively, projection onto the column and row spaces of \( \mathbf{X}_0 \). It can be shown that \( \mathcal{P}_\Phi(\mathbf{X}) = \mathbf{P}_U \mathbf{X} + \mathbf{XP}_V - \mathbf{P}_U \mathbf{XP}_V \), while the projection onto the complement subspace is \( \mathcal{P}_{\Phi^\perp}(\mathbf{X}) = (\mathbf{I} - \mathbf{P}_U)\mathbf{X}(\mathbf{I} - \mathbf{P}_V) \). In addition, the following identities
\[
\langle \mathcal{P}_\Phi(\mathbf{X}), \mathcal{P}_\Phi(\mathbf{Y}) \rangle = \langle \mathcal{P}_\Phi(\mathbf{X}), \mathbf{Y} \rangle = \langle \mathbf{X}, \mathcal{P}_\Phi(\mathbf{Y}) \rangle
\]
of orthogonal projection operators such as \( \mathcal{P}_\Phi(\cdot) \), will be invoked throughout the paper.

### A. Incoherence measures

Building on Lemma 1, alternative sufficient conditions are derived here to ensure local identifiability. To quantify the overlap between \( \Phi \) and \( \Omega_R \), consider the incoherence parameter
\[
\mu(\Omega_R, \Phi) = \max_{\mathbf{Z} \in \Omega_R \setminus \{0\}} \frac{\|\mathcal{P}_\Phi(\mathbf{Z})\|_F}{\|\mathbf{Z}\|_F}.
\]
for which it holds that \( \mu(\Omega_R, \Phi) \in [0,1] \). The lower bound is achieved when \( \Phi \) and \( \Omega_R \) are orthogonal, while the upper bound is attained when \( \Phi \cap \Omega_R \) contains a nonzero element. Assuming \( \Phi \cap \Omega_R = \{0_{L \times T}\} \), then \( \mu(\Omega_R, \Phi) < 1 \) represents the cosine of the angle between \( \Phi \) and \( \Omega_R \) [17]. From Lemma 1, it appears that \( \mu(\Omega_R, \Phi) < 1 \) guarantees \( \Phi \cap \Omega_R = \{0_{L \times T}\} \). As it will become clear later on, tighter conditions on \( \mu(\Omega_R, \Phi) \) will prove instrumental to guarantee exact recovery of \( \{\mathbf{X}_0, \mathbf{A}_0\} \) by solving (P1).
To measure the incoherence among subsets of columns of $\mathbf{R}$, which is tightly related to the second condition in Lemma 1, the restricted isometry constants (RICs) come handy [12]. The constant $\delta_k(\mathbf{R})$ measures the extent to which a $k$-subset of columns of $\mathbf{R}$ behaves like an isometry. It is defined as the smallest value satisfying

$$c(1 - \delta_k(\mathbf{R})) \leq \frac{\|\mathbf{Ru}\|^2}{\|\mathbf{u}\|^2} \leq c(1 + \delta_k(\mathbf{R}))$$

(4)

for every $\mathbf{u} \in \mathbb{R}^F$ with $\|\mathbf{u}\|_0 \leq k$ and for some positive normalization constant $c < 1$ [12]. For later use, introduce $\theta_{s_1,s_2}(\mathbf{R})$ which measures ‘how orthogonal’ are the subspaces generated by two disjoint column subsets of $\mathbf{R}$, with cardinality $s_1$ and $s_2$. Formally, $\theta_{s_1,s_2}(\mathbf{R})$ is the smallest value that satisfies

$$|\langle \mathbf{R}_1 \mathbf{u}_1, \mathbf{R}_2 \mathbf{u}_2 \rangle| \leq c \theta_{s_1,s_2}(\mathbf{R}) \|\mathbf{u}_1\| \|\mathbf{u}_2\|$$

(5)

for every $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^F$, where $\text{supp}(\mathbf{u}_1) \cap \text{supp}(\mathbf{u}_2) = \emptyset$ and $\|\mathbf{u}_1\|_0 \leq s_1, \|\mathbf{u}_2\|_0 \leq s_2$. The normalization constant $c$ plays the same role as in $\delta_k(\mathbf{R})$. A wide family of matrices with small RICs have been introduced in e.g., [12].

All the elements are now in place to state this section’s main result.

**Proposition 1:** Assume that each column of $\mathbf{A}_0$ contains at most $k$ nonzero elements. If $\mu(\Omega_R, \Phi) < 1$ and $\delta_k(\mathbf{R}) < 1$, then $\Omega_R \cap \Phi = \{0_{L \times T}\}$ and $\mathbf{R} \mathbf{H} \neq 0_{L \times T}, \mathbf{H} \in \Omega \setminus \{0_{F \times T}\}$.

**Proof:** Suppose the intersection is non trivial, meaning that there exists nonzero matrices $\mathbf{H} \in \Omega$ and $\mathbf{W}_1' + \mathbf{W}_2' \in \Phi$ satisfying $\mathbf{R} \mathbf{H} = \mathbf{U} \mathbf{W}_1' + \mathbf{W}_2'$. Vectorizing the last equation and relying on the identity $\text{vec}(\mathbf{AXB}) = (\mathbf{B}^\prime \otimes \mathbf{A})\text{vec}(\mathbf{X})$, one obtains a linear system of equations

$$[\mathbf{I}_T \otimes \mathbf{R} - \mathbf{I}_T \otimes \mathbf{U} - \mathbf{V} \otimes \mathbf{I}_L] \mathbf{w} = 0_{LT}$$

(6)

where $\mathbf{w} := [\text{vec}(\mathbf{H})' \text{vec}(\mathbf{W}_1') \text{vec}(\mathbf{W}_2')]'$. Define an $LT \times FT$ matrix $\mathbf{C}_1 := \mathbf{I}_T \otimes \mathbf{R}$ and the $LT \times (L+T)r$ matrix $\mathbf{C}_2 := [-\mathbf{I}_T \otimes \mathbf{U} - \mathbf{V} \otimes \mathbf{I}_L]$. The corresponding coefficients are $\mathbf{w}_1 := \text{vec}(\mathbf{H})$ and $\mathbf{w}_2 := [\text{vec}(\mathbf{W}_1') \text{vec}(\mathbf{W}_2')]'$. Then, (6) implies there exists a $\mathbf{w}_1 \neq 0_{FT}$ such that $\mathbf{C}_1 \mathbf{w}_1 + \mathbf{C}_2 \mathbf{w}_2 = 0_{LT}$.

Consider two cases: i) $\mathbf{w}_2 = 0_{r(L+T)}$, and ii) $\mathbf{w}_2 \neq 0_{r(L+T)}$. Under i) $\mathbf{C}_1 \mathbf{w}_1 = 0_{LT}$, and thus $\mathbf{R} \mathbf{w}_1^{(i)} = 0$ for some nonzero $\mathbf{w}_1^{(i)}$ with $i \in \{1, 2, \ldots, T\}$ where $\mathbf{w}_1 = [\mathbf{w}_1^{(1)} \ldots \mathbf{w}_1^{(T)}]$. Therefore, if $\|\mathbf{w}_1^{(i)}\|_0 \leq k$, $\delta_k(\mathbf{R}) < 1$ implies that $\mathbf{w}_1^{(i)} = 0_{LT}$, which is a contradiction. For ii) $\mu(\Omega_R, \Phi) < 1$ implies that there is no $\mathbf{w}_1$ with $\text{supp}(\mathbf{w}_1) \subseteq \text{supp}(\text{vec}(\mathbf{A}_0))$ and $\mathbf{w}_2 \in \mathbb{R}^{(L+T)r}$ such that $\mathbf{C}_1 \mathbf{w}_1 + \mathbf{C}_2 \mathbf{w}_2 = 0_{FT}$, since otherwise $|\langle \mathbf{C}_1 \mathbf{w}_1, \mathbf{C}_2 \mathbf{w}_2 \rangle| = \|\mathbf{C}_1 \mathbf{w}_1\| \|\mathbf{C}_2 \mathbf{w}_2\|$ which leads to $\mu(\Omega_R, \Phi) = 1$. \hfill \blacksquare

**III. Exact Recovery via Convex Optimization**

In addition to $\mu(\Omega_R, \Phi)$, there are other incoherence measures which play an important role in the conditions for exact recovery. Consider a feasible solution $\{\mathbf{X}_0 + a_{ij} \mathbf{R}_{e_i e_j}', \mathbf{A}_0 - a_{ij} e_i e_j'\}$, where $(i, j) \notin
Theorem 1: Consider given matrices $Y \in \mathbb{R}^{L \times T}$ and $R \in \mathbb{R}^{L \times F}$ obeying $Y = X_0 + RA_0 = U \Sigma V' + RA_0$, with $r := \text{rank}(X_0)$ and $s := \|A_0\|_0$. Assume that every row and column of $A_0$ has at most $k$ nonzero elements, and that $R$ has orthonormal rows. If the following conditions

I) \quad (1 - \mu(\Phi, \Omega_R))^2 (1 - \delta_k(R)) > \omega_{\max}; \quad \text{and}

II) \quad (1 + \alpha_{\max}) \left( \frac{1 + \beta_{\max}}{1 - \beta_{\max}} \right) \xi_R(U, V) \sqrt{s} + \mu(\Phi, \Omega_R)(1 + \delta_k(R))^{1/2} (1 + \alpha_{\max}) \sqrt{r} < 1
hold, where

$$\omega_{\text{max}} := \theta_{1,1}(R)[\sqrt{2k} + s\gamma^2(V)] + (1 + \delta_1(R)) \left[\sqrt{2k\gamma^2_R(U)} + k\gamma^2(V) + s\gamma^2_R(U)\gamma^2(V)\right]$$

$$\alpha_{\text{max}} := \left[\frac{1}{c(1 - \delta_k(R))(1 - \mu(\Phi, \Omega))^2} - 1\right]^{1/2}$$

$$\beta_{\text{max}} := \frac{1}{(1 - \mu(\Omega_R, \Phi))^2(1 - \delta_k(R))\omega_{\text{max}} - 1}$$

then there exists $\lambda > 0$ for which the convex program (P1) exactly recovers $\{X_0, A_0\}$.

Note that I) alone is already more stringent than the pair of conditions $\mu(\Omega_R, \Phi) < 1$ and $\delta_k(R) < 1$ needed for local identifiability (cf. Proposition 1). Satisfaction of the conditions in Theorem 1 hinges upon the values of the incoherence parameters $\mu(\Omega_R, \Phi), \gamma_R(U), \gamma(V), \xi_R(U, V)$, and the RICs $\delta_k(R)$ and $\theta_{1,1}(R)$. In particular, $\{\omega_{\text{max}}, \alpha_{\text{max}}, \beta_{\text{max}}\}$ are increasing functions of these parameters, and it is readily observed from I) and II) that the smaller $\{\omega_{\text{max}}, \alpha_{\text{max}}, \beta_{\text{max}}\}$ are, the more likely the conditions are met.

Furthermore, the incoherence parameters are increasing functions of the rank $r$ and sparsity level $s$. The RIC $\delta_k(R)$ is also an increasing function of $k$, the maximum number of nonzero elements per row/column of $A_0$. Therefore, for sufficiently small values of $\{r, s, k\}$, the sufficient conditions of Theorem 1 can be indeed satisfied.

It is worth noting that not only $s$, but also the position of the nonzero entries in $A_0$ plays an important role in satisfying I) and II). This is manifested through $k$, for which a small value indicates the entries of $A_0$ are sufficiently spread out, i.e., most entries do not cluster along a few rows or columns of $A_0$. Moreover, no restriction is placed on the magnitude of these entries, since as seen later on it is only the positions that affect optimal recovery via (P1).

**Remark 2 (Row orthonormality of $R$):** Assuming $RR' = I_L$ is equivalent to supposing that $R$ is full-rank. This is because for a full row-rank $R = U_R\Sigma_RV_R'$, one can pre-multiply both sides of (1) with $\Sigma_R^{-1}U_R'$ to obtain $\tilde{R} := V_R'$ with orthonormal rows.

**B. Induced recovery results for principal components pursuit and compressed sensing**

Before delving into the proof of the main result, it is instructive to examine how the sufficient conditions in Theorem 1 simplify for the subsumed PCP and CS problems. In PCP one has $R = I_L$, which implies $\Omega_R = \Omega$ and $\delta_k(R) = \theta_{1,1}(R) = 0$. To obtain sufficient conditions expressed only in terms of $\mu(\Phi, \Omega)$, one can borrow the coherence conditions of [10] and readily arrive at the following result.

**Corollary 1:** Consider given $Y \in \mathbb{R}^{L \times T}$ obeying $Y = X_0 + A_0 = U\Sigma V' + A_0$, with $r := \text{rank}(X_0)$ and $s := \|A_0\|_0$. Suppose the coherence conditions $\gamma(U) := \max_i \|P_U e_i\| \leq \sqrt{pr/L}$, $\gamma(V) \leq \sqrt{pr/T}$.
and \( \xi(U, V) := \|UV\|_\infty \leq \sqrt{\rho r/LT} \) hold for some positive constant \( \rho \). If \( \mu(\Phi, \Omega) \) is sufficiently small such that the following conditions
\[
\begin{align*}
I) & \quad 0 < \mu(\Phi, \Omega) < 1 - \sqrt{\omega_{\max}} \quad \text{and} \\
II) & \quad (1 + \alpha_{\max}) \sqrt{T} \left\{ \left( \frac{1+\beta_{\max}}{1-\beta_{\max}} \right) \sqrt{\frac{\rho r}{LT}} + \mu(\Phi, \Omega) \right\} < 1
\end{align*}
\]
hold, where
\[
\omega_{\max} := \rho rk \left( \frac{1}{L} + \frac{1}{T} \right)
\]
\[
\alpha_{\max} := \left[ \frac{1}{(1 - \mu(\Phi, \Omega))^2} - 1 \right]^{1/2}
\]
\[
\beta_{\max} := \frac{1}{(1 - \mu(\Phi, \Omega))^2(\omega_{\max}^{-1}) - 1}
\]
then there exists \( \lambda > 0 \) for which the convex program (P1) with \( R = I_L \) exactly recovers \( \{X_0, A_0\} \).

In Section V, random matrices \( \{X_0, A_0, R\} \) drawn from natural ensembles are shown to satisfy I) and II) with high probability. In this case, it is possible to arrive at simpler conditions (depending only on \( r, s, \) and the matrix dimensions) for exact recovery in the context of PCP; see Remark 6. Corollary 1, on the other hand, offers general conditions stemming from a purely deterministic approach.

In the CS setting one has \( X_0 = 0_{L \times T} \), which implies \( \mu(\Phi, \Omega_R) = \xi_R(U, V) = \gamma_R(U) = \gamma(V) = 0 \). As a result, Theorem 1 simply boils down to a RIC-dependent sufficient condition for the exact recovery of \( A_0 \) as stated next.

**Corollary 2:** Consider given matrices \( Y \in \mathbb{R}^{L \times T} \) and \( R \in \mathbb{R}^{L \times F} \) obeying \( Y = RA_0 \). Assume that the number of nonzero elements per column of \( A_0 \) does not exceed \( k \). If
\[
\delta_k(R) + k \theta_{1,1}(R) < 1 \tag{7}
\]
holds, then (P1) with \( X = 0_{L \times T} \) exactly recovers \( A_0 \).

To place (7) in context, consider normalizing the rows of \( R \). For such a compression matrix it is known that \( \delta_k(R) \leq (k - 1) \theta_{1,1}(R) \), see e.g., [31]. Using this bound together with (7), one arrives at the stricter condition \( k < \frac{1}{2} \left( 1 + \theta_{1,1}^{-1}(R) \right) \). This last condition is identical to the one reported in [19], which guarantees the success of \( \ell_1 \)-norm minimization in recovering sparse solutions to under-determined systems of linear equations. The conditions have been improved in recent works; see e.g., [31] and references therein.

**IV. PROOF OF THE MAIN RESULT**

In what follows, conditions are first derived under which \( \{X_0, A_0\} \) is the unique optimal solution of (P1). In essence, these conditions are expressed in terms of certain dual certificates. Then, Section IV-B
deals with the construction of a valid dual certificate.

A. Unique optimality conditions

Recall the nonsmooth optimization problem (P1), and its Lagrangian

$$\mathcal{L}(X, A, M) = \|X\|_* + \lambda \|A\|_1 + \langle M, Y - X - RA \rangle$$

(8)

where \(M \in \mathbb{R}^{L \times T}\) is the matrix of dual variables (multipliers) associated with the constraint in (P1). From the characterization of the subdifferential for nuclear- and \(\ell_1\)-norm (see e.g., [8]), the subdifferential of the Lagrangian at \(\{X_0, A_0\}\) is given by (recall that \(X_0 = U\Sigma V')

$$\partial_X \mathcal{L}(X_0, A_0, M) = \{UV' + W - M : \|W\| \leq 1, \mathcal{P}_\Phi(W) = 0_{L \times T}\}$$

(9)

$$\partial_A \mathcal{L}(X_0, A_0, M) = \{\lambda \text{sign}(A_0) + \lambda F - R'M : \|F\|_\infty \leq 1, \mathcal{P}_\Omega(F) = 0_{F \times T}\}.$$  

(10)

The optimality conditions for (P1) assert that \(\{X_0, A_0\}\) is an optimal (not necessarily unique) solution if and only if

$$0_{F \times T} \in \partial_A \mathcal{L}(X_0, A_0, M) \text{ and } 0_{L \times T} \in \partial_X \mathcal{L}(X_0, A_0, M).$$

This can be shown equivalent to finding the pair \(\{W, F\}\) that satisfies: i) \(\|W\| \leq 1, \mathcal{P}_\Phi(W) = 0_{L \times T}\); ii) \(\|F\|_\infty \leq 1, \mathcal{P}_\Omega(F) = 0_{F \times T}\); and iii) \(\lambda \text{sign}(A_0) + \lambda F = R'(UV' + W)\). In general, i)-iii) may hold for multiple solution pairs. However, the next lemma asserts that a slight tightening of the optimality conditions i)-iii) leads to a unique optimal solution for (P1). See Appendix A for a proof.

**Lemma 2:** Assume that each column of \(A_0\) contains at most \(k\) nonzero elements, as well as \(\mu(\Omega_R, \Phi) < 1\) and \(\delta_k(R) < 1\). If there exists a dual certificate \(\Gamma \in \mathbb{R}^{L \times T}\) satisfying

C1) \(\mathcal{P}_\Phi(\Gamma) = UV'\)

C2) \(\mathcal{P}_\Omega(R\Gamma) = \lambda \text{sign}(A_0)\)

C3) \(\|\mathcal{P}_\Phi(\Gamma)\| < 1\)

C4) \(\|\mathcal{P}_\Omega(\Gamma)\|_\infty < \lambda\)

then \(\{X_0, A_0\}\) is the unique optimal solution of (P1).

The remainder of the proof deals with the construction of a dual certificate \(\Gamma\) that meets C1)-C4). To this end, tighter conditions [I] and II) in Theorem 1] for the existence of \(\Gamma\) are derived in terms of the incoherence parameters and the RICs. For the special case \(R = I_L\), the conditions in Lemma 2 boil down to those in [14, Prop. 2] for PCP. However, the dual certificate construction techniques used in [14] do not carry over to the setting considered here, where a compression matrix \(R\) is present.
B. Dual certificate construction

Condition C1) in Lemma 2 implies that $\Gamma = UV' + (I - P_U)X(I - P_V)$, for arbitrary $X \in \mathbb{R}^{LxT}$ (cf. Remark 1). Upon defining $Z := R'(I - P_U)X(I - P_V)$ and $B_\Omega := \lambda \text{sign}(A_0) - \mathcal{P}_\Omega(R'UV')$, C1) and C2) are equivalent to $\mathcal{P}_\Omega(Z) = B_\Omega$.

To express $\mathcal{P}_\Omega(Z) = B_\Omega$ in terms of the unrestricted matrix $X$, first vectorize $Z$ to obtain $\text{vec}(Z) = [(I - P_V) \otimes R'(I - P_U)] \text{vec}(X)$. Define $A := (I - P_V) \otimes R'(I - P_U)$ and an $s \times LT$ matrix $A_\Omega$ formed with those $s$ rows of $A$ associated with those elements in supp($A_0$). Likewise, define $A_{\Omega^c}$ which collects the remaining rows from $A$ such that $A = \Pi[A'_{\Omega}, A'_{\Omega^c}]'$ for a suitable row permutation matrix $\Pi$. Finally, let $b_\Omega$ be the vector of length $s$ containing those elements of $B_\Omega$ with indices in supp($A_0$). With these definitions, C1) and C2) can be expressed as $A_\Omega \text{vec}(X) = b_\Omega$.

To upper-bound the left-hand side of C3) in terms of $X$, use the assumption $RR' = I_L$ to arrive at

$$ \|\mathcal{P}_{\Omega^c}(\Gamma)\| = \|R'(I - P_U)X(I - P_V)\| \leq \|R'(I - P_U)X(I - P_V)\|_F = \|A\text{vec}(X)\|.$$ 

Similarly, the left-hand side of C4) can be bounded as

$$ \|\mathcal{P}_{\Omega^c}(R'T)\|_\infty = \|\mathcal{P}_{\Omega^c}(Z) + \mathcal{P}_{\Omega^c}(R'UV')\|_\infty \leq \|\mathcal{P}_{\Omega^c}(Z)\|_\infty + \|\mathcal{P}_{\Omega^c}(R'UV')\|_\infty$$ 

$$ = \|A_{\Omega^c}\text{vec}(X)\|_\infty + \|\mathcal{P}_{\Omega^c}(R'UV')\|_\infty.$$ 

In a nutshell, if one can find $X \in \mathbb{R}^{LxT}$ such that

1. $A_{\Omega^c}\text{vec}(X) = b_\Omega$
2. $\|A\text{vec}(X)\| < 1$
3. $\|A_{\Omega^c}\text{vec}(X)\|_\infty + \|\mathcal{P}_{\Omega^c}(R'UV')\|_\infty < \lambda$

hold for some positive $\lambda$, then C1)-C4) would be satisfied as well.

The final steps of the proof entail: i) finding an appropriate candidate solution $\hat{X}$ such that c1) holds; and ii) deriving conditions in terms of the incoherence parameters and RICs that guarantee $\hat{X}$ meets the required bounds in c2) and c3) for a range of $\lambda$ values. The following lemma is instrumental to accomplishing i), and its proof can be found in Appendix B.

**Lemma 3:** Assume that each column of $A_0$ contains at most $k$ nonzero elements, as well as $\mu(\Omega_R, \Phi) < 1$ and $\delta_k(R) < 1$. Then matrix $A_{\Omega}$ has full row rank, and its minimum singular value is bounded below as

$$ \sigma_{\min}(A'_{\Omega}) \geq c^{1/2}(1 - \delta_k(R))^{1/2}(1 - \mu(\Phi, \Omega_R)).$$
According to Lemma 3, the least-norm (LN) solution \( \hat{X}_{LN} := \arg\min_X \{ \|X\|_F^2 : A_\Omega \text{vec}(X) = b_\Omega \} \) exists, and is given by

\[
\text{vec}(\hat{X}_{LN}) = A_\Omega' (A_\Omega A_\Omega')^{-1} b_\Omega.
\] (11)

**Remark 3 (Candidate dual certificate):** From the arguments at the beginning of this section, the candidate dual certificate is \( \hat{\Gamma} := UV' + (I - P_U)\hat{X}_{LN}(I - P_V) \).

The LN solution is an attractive choice, since it facilitates satisfying \( c_2 \) and \( c_3 \) which require norms of \( \text{vec}(X) \) to be small. Substituting the LN solution (11) into the left hand side of \( c_2 \) yields (define \( Q := A_\Omega A_\Omega' (A_\Omega A_\Omega')^{-1} \) for notational brevity)

\[
\|A \text{vec}(\hat{X}_{LN})\| = \left\| \left( \begin{array}{c} A_\Omega \\ A_\Omega' \end{array} \right) \left( \begin{array}{c} A_\Omega' (A_\Omega A_\Omega')^{-1} b_\Omega \\ I \end{array} \right) \right\| \leq (1 + \|Q\|) \|b_\Omega\|.
\] (12)

Moreover, substituting (11) in the left hand side of \( c_3 \) results in

\[
\|Qb_\Omega\|_\infty + \|P_{\Omega^c} (R'UV')\|_\infty \leq \|Q\|_{\infty,\infty} \|b_\Omega\|_\infty + \|P_{\Omega^c} (R'UV')\|_\infty.
\] (13)

Next, upper-bounds are obtained for \( \|Q\| \) and \( \|Q\|_{\infty,\infty} \); see Appendix C for a proof.

**Lemma 4:** Assume that each column and row of \( A_0 \) contains at most \( k \) nonzero elements. If \( \mu(\Omega_R, \Phi) < 1 \) and \( \delta_k(R) < 1 \) hold, then

\[
\|Q\| \leq \alpha_{\text{max}} := \left[ \frac{1}{c(1 - \delta_k(R))(1 - \mu(\Omega_R, \Phi))^2} - 1 \right]^{1/2}.
\]

If the tighter condition \( I \) holds instead, then

\[
\|Q\|_{\infty,\infty} \leq \beta_{\text{max}} := \frac{\omega_{\text{max}}}{(1 - \mu(\Omega_R, \Phi))^2(1 - \delta_k(R)) - \omega_{\text{max}}}.
\]

Going back to (12)-(13), note that \( \|B_\Omega\|_\infty = \|b_\Omega\|_\infty \) and \( \|B_\Omega\|_F = \|b_\Omega\|_F \), which can be respectively upper-bounded as

\[
\|B_\Omega\|_\infty = \|\lambda \text{sign}(A_0) - P_{\Omega}(R'UV')\|_\infty \leq \lambda + \|P_{\Omega}(R'UV')\|_\infty
\] (14)

\[
\|B_\Omega\|_F = \|\lambda \text{sign}(A_0) - P_{\Omega}(R'UV')\|_F \leq \lambda \sqrt{s} + \|P_{\Omega}(R'UV')\|_F.
\] (15)
Finally, \( \|P_\Omega(R'UV')\|_F \) itself can be bounded above as

\[
\|P_\Omega(R'UV')\|_F^2 = |\langle P_\Omega(R'UV'), P_\Omega(R'UV') \rangle| \overset{(a)}{=} |\langle R'UV', P_\Omega(R'UV') \rangle|
\]

\[
= |\langle UV', RP_\Omega(R'UV') \rangle| \overset{(b)}{=} |\langle P_\Phi(UV'), P_\Phi(RP_\Omega(R'UV')) \rangle|
\]

\[
\overset{(c)}{\leq} \|P_\Phi(UV')\|_F \|P_\Phi(RP_\Omega(R'UV'))\|_F
\]

\[
\overset{(d)}{\leq} \|UV'\|_F \mu(\Phi, \Omega_r) \|RP_\Omega(R'UV')\|_F
\]

\[
\overset{(e)}{\leq} \sqrt{\mu(\Phi, \Omega_r)} c^{1/2}(1 + \delta_k(R))^{1/2} \|P_\Omega(R'UV')\|_F
\]

(16)

where (a) is due to (2), (b) follows because \( UV' \in \Phi \) (thus \( \mathcal{P}_\Phi(UV') = UV' \)) and from the property in (2). Moreover, (c) is a direct result of the Cauchy-Schwarz inequality, while (d) and (e) come from (3) and (4), respectively, and the assumption that number of nonzero elements per column of \( A_0 \) does not exceed \( k \). All in all, \( \|P_\Omega(R'UV')\|_F \leq \sqrt{\mu(\Phi, \Omega_R)} c^{1/2}(1 + \delta_k(R))^{1/2} \) and (15) becomes

\[
\|B_\Omega\|_F \leq \lambda \sqrt{s} + \sqrt{\mu(\Phi, \Omega_r)} c^{1/2}(1 + \delta_k(R))^{1/2}.
\]

(17)

Upon substituting (14), (17) and the bounds in Lemma 4 into (12) and (13), one finds that c2) and c3) hold if there exists \( \lambda > 0 \) such that

\[
(1 + \alpha_{\max}) \left[ \lambda \sqrt{s} + \sqrt{\mu(\Omega_R, \Phi)} c^{1/2}(1 + \delta_k(R))^{1/2} \right] < 1
\]

(18a)

\[
\beta_{\max} \left( \lambda + \|P_\Omega(R'UV')\|_\infty \right) + \|P_{\Omega^+}(R'UV')\|_\infty < \lambda
\]

(18b)

hold. Recognizing that \( \xi_R(U, V) = \max\{\|P_\Omega(R'UV')\|_\infty, \|P_{\Omega^+}(R'UV')\|_\infty\} \), the left-hand side of (18b) can be further bounded. After straightforward manipulations, one deduces that conditions (18a) and (18b) are satisfied for \( \lambda \in (\lambda_{\min}, \lambda_{\max}) \), where

\[
\lambda_{\min} := \left( \frac{1 + \beta_{\max}}{1 - \beta_{\max}} \right) \xi_R(U, V)
\]

\[
\lambda_{\max} := \frac{1}{\sqrt{s}} \left[ (1 + \alpha_{\max})^{-1} - \sqrt{\mu(\Omega_R, \Phi)} c^{1/2}(1 + \delta_k(R))^{1/2} \right].
\]

Clearly, it is still necessary to ensure \( \lambda_{\max} > \lambda_{\min} \) so that the LN solution (11) meets the requirements c1)-c3) [equivalently, \( \hat{R} \) in Remark 3 satisfies C1)-C4) from Lemma 2]. Condition \( \lambda_{\max} > \lambda_{\min} \) is equivalent to II) in Theorem 1, and the proof is now complete.

**Remark 4 (Satisfiability):** From a high-level vantage point, Theorem 1 asserts that (P1) recovers \( \{ X_0, A_0 \} \) when the components \( X_0 \) and \( RA_0 \) are sufficiently incoherent, and the compression matrix \( R \) has good restricted isometry properties. It should be noted though, that given a triplet \( \{ X_0, A_0, R \} \) in general one cannot directly check whether the sufficient conditions I) and II) hold, since e.g., \( \delta_k(R) \) is NP-hard to
compute [12]. This motivates finding a class of (possibly random) matrices \( \{X_0, A_0, R\} \) satisfying I) and II), the subject dealt with next.

V. Matrices Satisfying the Conditions for Exact Recovery

This section investigates triplets \( \{X_0, A_0, R\} \) satisfying the conditions of Theorem 1, henceforth termed admissible matrices. Specifically, it will be shown that low-rank, sparse, and compression matrices drawn from certain random ensembles satisfy the sufficient conditions of Theorem 1 with high probability.

A. Uniform sparsity model

Matrix \( A_0 \) is said to be generated according to the uniform sparsity model, when drawn uniformly at random from the collection of all matrices with support size \( s \). There is no restriction on the amplitude of the nonzero entries. An attractive property of this model is that it guarantees (with high probability) that no single row or column will monopolize most nonzero entries of \( A_0 \), for sufficiently large \( A_0 \) and appropriate scaling of the sparsity level. This property is formalized in the following lemma (for simplicity in exposition it is henceforth assumed that that \( A_0 \) is a square matrix, i.e., \( F = T \)).

**Lemma 5:** [14] If \( A_0 \in \mathbb{R}^{F \times F} \) is generated according to the uniform sparsity model with \( \|A_0\|_0 = s \), then the maximum number \( k \) of nonzero elements per column or row of \( A_0 \) is bounded as

\[
k \leq \frac{s}{F} \log(F)
\]

with probability higher than \( 1 - O(F^{-\zeta}) \), for \( s = O(\zeta F) \).

In practice, it is simpler to work with the Bernoulli model that specifies \( \text{supp}(A_0) = \{(i,j) : b_{i,j} = 1\} \), where \( \{b_{i,j}\} \) are independent and identically distributed (i.i.d.) Bernoulli random variables taking value one with probability \( \pi := s/F^2 \), and zero with probability \( 1 - \pi \). There are three important observations regarding the Bernoulli model. First, \( |\text{supp}(A_0)| \) is a random variable, whose expected value is \( s \) and matches the uniform sparsity model. Second, arguing as in [10, Lemma 2.2] one can claim that if (P1) exactly recovers \( \{X_0, A_0\} \) from data \( Y = X_0 + RA_0 \), it will also exactly recover \( \{X_0, \hat{A}_0\} \) from \( \hat{Y} = X_0 + R\hat{A}_0 \) when \( \text{supp}(\hat{A}_0) \subseteq \text{supp}(A_0) \) and the nonzero entries coincide. Third, following the logic of [11, Section II.C] one can prove that the failure rate\(^1\) for the uniform sparsity model is bounded by twice the failure rate corresponding to the Bernoulli model. As a result, any recovery guarantee established for the Bernoulli model holds for the uniform sparsity model as well.

\(^1\)The failure rate is defined as \( \Pr(\hat{A} \neq A_0) \), where \( \hat{A} \) is the solution of (P1).
In addition to the bound for $k$ in Lemma 5, the Bernoulli model can be used to bound $\mu(\Phi, \Omega_R)$ in terms of the incoherence parameters $\{\gamma_R(U), \gamma(V)\}$ and the RIC $\delta_k(R)$. For a proof, see Appendix D.

**Lemma 6**: Let $\Lambda := \sqrt{c(1 + \delta_1(R))} \left[ \gamma^2_R(U) + \gamma^2(V) \right]^{1/2}$ and $n := \max\{L, F\}$. Suppose $A_0 \in \mathbb{R}^{F \times F}$ is generated according to the Bernoulli model with $\text{Pr}(b_{i,j} = 1) = \pi$, and $RR' = I_L$. Then, there exist positive constants $C$ and $\tau$ such that
\[
\mu(\Phi, \Omega_R) \leq \sqrt{c^{-1}(1 - \delta_k(R))^{-1}} \left[ C\Lambda \sqrt{\log(LF)/\pi} + \tau \Lambda \log(n) + 1 \right]^{1/2}
\] (19)
holds with probability at least $1 - n^{-C\pi \Lambda \tau}$ if $\delta_k(R)$ and the right-hand side of (19) do not exceed one.\(^2\)

Consider (19) when $\Lambda$ is small enough so that the quantity inside the square brackets is close to one. One obtains $\mu(\Phi, \Omega_R) \leq \sqrt{c^{-1}(1 - \delta_k(R))^{-1}}$, which reduces to the bound $\mu(\Phi, \Omega) \leq \sqrt{\pi}$ derived in [10, Section 2.5] for the special case $R = I_L$. Hence, the price paid in terms of coherence increase due to $R$ is roughly $\sqrt{c^{-1}(1 - \delta_k(R))^{-1}} > 1$. As expected, (19) also shows that for $R$ with small RICs the incoherence between subspaces $\Phi$ and $\Omega_R$ becomes smaller, and identifiability is more likely.

The result in Lemma 6 allows one to ‘eliminate’ $\mu(\Phi, \Omega_R)$ from the sufficient conditions in Theorem 1, which can thus be expressed only in terms of $\{\gamma_R(U), \gamma(V), \xi_R(U, V)\}$ and the RICs of $R$. In the following sections, random low-rank and compression matrices giving rise to small incoherence parameters and RICs are described.

### B. Random orthogonal model

Among other implications, matrices $X_0$ and $R$ with small $\gamma_R(U)$ and $\xi_R(U, V)$ are such that the columns of $R$ (approximately) fall outside the column space of $X_0$. From a design perspective, this suggests that the choice of an admissible $X_0$ (or in general an ensemble of low-rank matrices) should take into account the structure of $R$, and vice versa. However, in the interest of simplicity one could seek conditions dealing with $X_0$ and $R$ separately, that still ensure $\gamma_R(U)$ and $\xi_R(U, V)$ are small. This way one can benefit from the existing theory on incoherent low-rank matrices developed in the context of matrix completion [9], and matrices with small RICs useful for CS [11], [31]. Admittedly, the price paid is in terms of stricter conditions that will reduce the set of admissible matrices.

In this direction, the next lemma bounds $\gamma_R(U)$ and $\xi_R(U, V)$ in terms of $\gamma(U) := \max_i \|P_U e_i\|$, $\gamma(V)$ and $\delta_k(R)$.

\(^2\)Even though one has $n = F$ and $\pi = s/F^2$ in the problem studied here, Lemma 6 is stated using $n$ and $\pi$ to retain generality.
Lemma 7: If $\eta(R) := \max_i \|Re_i\|_1 / \|Re_i\|$, it then holds that

$$\gamma_R(U) \leq \eta(R)\gamma(U) \quad (20)$$

$$\xi_R(U, V) \leq \sqrt{c(1 + \delta_1(R))\eta(R)\gamma(U)\gamma(V)} \quad (21)$$

Proof: Starting from the definition

$$\gamma_R(U) = \max_i \|P_U Re_i\| / \|Re_i\| = \max_i \|P_U \sum \epsilon_i e_i' Re_i\| / \|Re_i\| \leq a \leq \max_i \sum \|P_U e_i\| / \|Re_i\| \leq b \leq \gamma(U) \max_i \|Re_i\|_1 / \|Re_i\| \quad (22)$$

where (a) follows from the Cauchy-Schwarz inequality, and (b) from the definition of $\gamma(U)$.

Likewise, applying the definition of $\xi_R(U, V)$ one obtains

$$\xi_R(U, V) = \max_{i, j} |e_i' R' UV e_j| \leq c \max_i \|U' Re_i\| \max_j \|V' e_j\| \leq \sqrt{c(1 + \delta_1(R))\gamma(U)\gamma(V)} \leq \sqrt{c(1 + \delta_1(R))\eta(R)\gamma(U)\gamma(V)} \quad (23)$$

where (c) follows from the Cauchy-Schwarz inequality, and (d) is due to (22).

The bounds (20) and (21) are proportional to $\gamma(U)$ and $\gamma(V)$. This prompts one to consider incoherent rank-$r$ matrices $X_0 = U\Sigma V'$ generated from the random orthogonal model, which is specified as follows. The singular vectors forming the columns of $U$ and $V$ are drawn uniformly at random from the collection of rank-$r$ partial isometries in $\mathbb{R}^{L \times r}$ and $\mathbb{R}^{F \times r}$, respectively. There is no need for $U$ and $V$ to be statistically independent, and no restriction in placed on the singular values in the diagonal of $\Sigma$. The adequacy of the random orthogonal model in generating incoherent low-rank matrices is justified by the following lemma (recall $T = F \geq L$).

Lemma 8: [14] If $X_0 = U\Sigma V' \in \mathbb{R}^{L \times F}$ is generated according to the random orthogonal model with rank($X_0$) = $r$, then

$$\max\{\gamma(U), \gamma(V)\} \leq \sqrt{\max\{r, \log(F)\}} / \sqrt{F}$$

with probability exceeding $1 - O(F^{-3} \log(F))$.

C. Random compressive matrices

With reference to Lemma 7 [cf. (20) and (21)], it is clear that an incoherent $X_0$ alone may not suffice to yield small $\gamma_R(U)$ and $\xi_R(U, V)$. In addition, $\eta(R) \in [1, \sqrt{L}]$ should be as close as possible to one.
This can be achieved e.g., when \( \mathbf{R} \) is sparse across each column. Note that the lower bound of unity is attained when \( \mathbf{R} \) has at most a single nonzero element per column, as it is the case when \( \mathbf{R} = \mathbf{I}_L \).

The aforementioned observations motivate considering block-diagonal compression matrices \( \mathbf{R} \in \mathbb{R}^{L \times F} \), consisting of blocks \( \{ \mathbf{R}_i \in \mathbb{R}^{\ell \times f} \} \) where \( \ell \leq f \). The number of blocks is \( n_b := F/f \) assuming that \( f \) divides \( F \). The \( i \)-th block is generated according to the bounded orthonormal model as follows; see e.g., [31]. For some positive constant \( K \), (deterministically) choose a unitary matrix \( \mathbf{\Psi} \in \mathbb{R}^{f \times f} \) with bounded entries

\[
\max_{(t,k) \in \mathcal{F} \times \mathcal{F}} |\mathbf{\Psi}_{t,k}| \leq K
\]

(24)

where \( \mathcal{F} := \{1, \ldots, f\} \). For each \( i = 1, \ldots, n_b \) form \( \mathbf{R}_i := \mathbf{\Theta}_{T^{(i)}} \mathbf{\Psi} \), where \( \mathbf{\Theta}_{T^{(i)}} := [\mathbf{e}_{t^{(i)}_1}, \ldots, \mathbf{e}_{t^{(i)}_s}]' \in \mathbb{R}^{\ell \times f} \) is a random row subsampling matrix that selects the rows of \( \mathbf{\Psi} \) indexed by \( T^{(i)} := \{ t^{(i)}_1, \ldots, t^{(i)}_s \} \subset \mathcal{F} \). In words, \( \mathbf{\Theta}_{T^{(i)}} \) is formed by those \( \ell \) rows of \( \mathbf{I}_f \) indexed by \( T^{(i)} \). The row indices in \( T^{(i)} \) are selected independently at random, with uniform probability \( 1/f \) from \( \mathcal{F} \). By construction, \( \mathbf{R}_i \mathbf{R}_i' = \mathbf{I}_\ell, i = 1, \ldots, n_b \), which ensures \( \mathbf{R} \mathbf{R}' = \mathbf{I}_L \) as required by Theorem 1. Most importantly, the next lemma states that such a construction for \( \mathbf{R}_i \) leads to small RICs with high probability; see e.g., [31] for the proof.

**Lemma 9:** [31] Let \( \mathbf{R}_i \in \mathbb{R}^{\ell \times f} \) be generated according to the bounded orthonormal model. If for some \( k_i \in [1, f], \epsilon \in (0, 1) \) and \( \mu \in (0, 1/2) \) the following condition

\[
\frac{\ell}{\log(10\ell)} \geq DK^2\mu^{-2}s\log^2(100k_i)\log(4f)\log(7\epsilon^{-1})
\]

holds where the constant \( D \leq 243, 150 \), then \( \delta_{k_i}(\mathbf{R}_i) \leq \mu \) with probability greater than \( 1 - \epsilon \).

Lemma 9 asserts that for large enough \( \ell \), the RIC \( \delta_{k_i}(\mathbf{R}_i) = \mathcal{O}((\log(100k_i))\log(10\ell)\log(4f)^{1/2}\sqrt{k_i/\ell}) \) with overwhelming probability.

Let \( k_i \) denote the maximum number of nonzero elements per ‘trimmed’ column of \( \mathbf{A}_0 \), the trimming being defined by the block of rows of \( \mathbf{A}_0 \) that are multiplied by \( \mathbf{R}_i \) when carrying out the product \( \mathbf{R}_i \mathbf{A}_0 \).

With these definitions, the RIC of \( \mathbf{R} \) is bounded as \( \delta_k(\mathbf{R}) \leq \max_{i} \{ \delta_{k_i}(\mathbf{R}_i) \} \). For \( \delta_k(\mathbf{R}) \) to be small as required by Theorem 1, the \( k_i \) should be much smaller than \( \ell \). Since \( \mathbf{A}_0 \) is generated according to the uniform sparsity model outlined in Section V-A, its nonzero elements are uniformly spread across rows and columns as per Lemma 5. Formally, it holds that \( k_i \leq \kappa := (s/Fn_b)\log(Fn_b) \) with probability \( 1 - \mathcal{O}([Fn_b]^{-\zeta}) \), where \( s = \| \mathbf{A}_0 \|_0 = \zeta Fn_b \); see e.g., [6]. Accordingly, from Lemma 9 one can infer that \( \delta_k(\mathbf{R}) = \mathcal{O}(100\kappa)\log(10\ell)\log(4f)^{1/2}\sqrt{\kappa/\ell} \) with high probability. Note that the bound for \( \delta_k(\mathbf{R}) \) depends on \( k \) through the variable \( s \) in \( \kappa \), and the relationship between \( s \) and \( k \) in Lemma 5. Regarding the RIC \( \theta_{1,1}(\mathbf{R}) \), it is bounded as \( \theta_{1,1}(\mathbf{R}) \leq \delta_2(\mathbf{R}) \) [12]. The normalization constant \( c \) in (4) and (5)
also equals $L/F \ll 1$. Recalling $\eta(R)$ (cf. Lemma 7) which was subject of the initial discussion in this section, it turns out that for such a construction of $R$ one obtains $\eta(R) \leq \sqrt{\ell} \ll \sqrt{L}$.

**Remark 5 (Row and column permutations):** The class of admissible compression matrices can be extended to matrices which are block diagonal up to row and column permutations. Let $\Pi_r$ ($\Pi_c$) denote, respectively, the row (column) permutation matrices that render $R$ block diagonal. Instead of (1) consider $\Pi_rX = \Pi_rX_0 + \Pi_rR\Pi_c\Pi'_cA_0$ and note that $\Pi_rX_0$ has the same coherence parameters as $X_0$, while $\Pi_rR\Pi_c$ has the same RICs as $R$, and $\Pi'_cA_0$ is still uniformly sparse. Thus, one can feed the transformed data to (P1) and since $\Pi_r$ and $\Pi_c$ are invertible, $\{X_0, A_0\}$ can be readily obtained from the recovered $\{\Pi_rX_0, \Pi'_cA_0\}$.

### D. Closing the loop

According to Lemmata 6 and 7, the incoherence parameters $\mu(\Phi, \Omega_R)$, $\gamma_R(U)$ and $\xi_R(U, V)$ which play a critical role toward exact decomposability in Theorem 1, can be upper-bounded in terms of $\gamma(U)$ and $\gamma(V)$. For random matrices $\{X_0, A_0, R\}$ drawn from specific ensembles, Lemmata 5, 8 and 9 assert that the incoherence parameters $\gamma(U)$ and $\gamma(V)$ as well as the RICs $\delta_k(R)$ and $\theta_{1,1}(R)$, are bounded above in terms of $r = \text{rank}(X_0)$, the degree of sparsity $s = \|A_0\|_0$, and the underlying matrix dimensions $L, F, \ell, f$. Alternative sufficient conditions for exact recovery, expressible only in terms of the aforementioned basic parameters, can be obtained by combining the bounds of this section along with I) and II) in Theorem 1. Hence, in order to guarantee that (P1) recovers $\{X_0, A_0\}$ with high probability and for given matrix dimensions, it suffices to check feasibility of a set of inequalities in $r$ and $s$.

To this end, focus on the asymptotic case where $L$ and $F$ are large enough, while $F = T$ for simplicity in exposition. Recall the conditions of Theorem 1 and suppose $\delta_k(R) = o(1)$ and $\mu(\Phi, \Omega_R) = o(1)$. This results in $\alpha_{\max} \approx \sqrt{F/L}$ and $\beta_{\max} \approx (\omega_{\max}^{-1} - 1)^{-1}$ when $L \ll F$. Satisfaction of I) and II) then requires $O(1)$ summands in the left-hand side of II), which gives rise to $\xi_R(U, V) = O(\sqrt{L/Fs})$, $\mu(\Phi, \Omega_R) = O(\sqrt{L/F})$, and $\omega_{\max} = O(1) < 1$. The latter which is indeed the bottleneck constraint can be satisfied if $\theta_{1,1}(R) = O(1/k)$, $\theta_{1,1}(R)\gamma^2(V) = O(1/s)$, $\gamma^2_R(U) = O(1/k)$, $\gamma^2(V) = O(1/k)$, and $\gamma^2_R(U)\gamma^2_R(V) = O(1/s)$. Utilizing the bounds in Lemmata 6–9 establishes the next corollary.

**Corollary 3:** Consider given matrices $Y \in \mathbb{R}^{L \times F}$ and $R \in \mathbb{R}^{L \times F}$ obeying $Y = X_0 + RA_0$, where $r := \text{rank}(X_0)$ and $s := \|A_0\|_0$. Suppose that: (i) $X_0$ is generated according to the random orthogonal model; (ii) $A_0$ is generated according to the uniform sparsity model; and (ii) $R = \text{diag}(R_1, \ldots, R_m)$ with blocks $R_i \in \mathbb{R}^{\ell \times f}$ generated according to the bounded orthogonal model. Define $\tilde{r} := \max\{r, \log(F)\}$. If $r$ and $s$ satisfy
i) $\tilde{r} \lesssim \frac{F}{\ell}$

ii) $s \lesssim \min \left\{ \frac{F^2}{\ell \log(\log(100 \ell \log^2(4F)))}, \frac{F^2}{\tilde{r}} \right\}$

iii) $s^{1/2} \log \left( 100 \frac{F}{\ell} \log \left( \frac{F^2}{\ell} \right) \right) \prec \left[ \frac{F^2 \ell \log(F^2/\ell) \log^2(F)}{\ell \log(F^2/\ell) \log^2(F)} \right]^{1/2}$

there is a positive $\lambda$ for which (P1) recovers $\{X_0, A_0\}$ with high probability.

Remark 6 (Principal components pursuit): For PCP where $R = I_L$ and $L = T$ (cf. Corollary 1), it can be readily verified that $s \min\{r, \log(L)\} = O(L^2/\log(L))$ suffices for exact recovery of $\{X_0, A_0\}$ by solving (P1). This guarantee is of course valid with high probability, provided $\{X_0, A_0, R\}$ are drawn from the random matrix ensembles outlined throughout this section. However, in the presence of the compression matrix $R$ more stringent conditions are imposed on the rank and sparsity level, as stated in Corollary 3. This is mainly because of the dominant summand $\sqrt{2k + s\gamma^2(V)}\theta_{1,1}(R)$ in $\omega_{\max}$ (cf. Theorem 1), which limits the extent to which $r$ and $s$ can be increased. If the correlation between any two columns of $R$ is small, then higher rank and less sparse matrices can be exactly recovered.

VI. ALGORITHMS

This section deals with iterative algorithms to solve the non-smooth convex optimization problem (P1).

A. Accelerated proximal gradient (APG) algorithm

The class of accelerated proximal gradient algorithms were originally studied in [29], [30], and they have been popularized for $\ell_1$-norm regularized regression; mostly due to the success of the fast iterative shrinkage-thresholding algorithm (FISTA) [2]. Recently, APG algorithms have been applied to matrix-valued problems such as those arising with nuclear-norm regularized estimators for matrix completion [36], and for (stable) PCP [24], [40]. APG algorithms offer several attractive features, most notably a convergence rate guarantee of $O(1/\sqrt{\epsilon})$ iterations to return an $\epsilon-$optimal solution. In addition, APG algorithms are first-order methods that scale nicely to high-dimensional problems arising with large networks.

The algorithm developed here builds on the APG iterations in [24], proposed to solve the stable PCP problem. One can relax the equality constraint in (P1) and instead solve

$$\min_S \left\{ \nu \|X\|_* + \nu \lambda \|A\|_1 + \frac{1}{2} \|Y - X - RA\|_F^2 \right\}$$

with $S := [X', A']'$, where the least-square term penalizes violations of the equality constraint, and $\nu > 0$ is a penalty coefficient. When $\nu$ approaches zero, (P2) achieves the optimal solution of (P1) [3]. The gradient of $f(S) := \frac{1}{2} \|Y - X - RA\|_F^2$ is Lipschitz continuous with a (minimum) Lipschitz constant $L_f = \lambda_{\max}([I_L \ R]/[I_L \ R])$, i.e., $\|\nabla f(S_1) - \nabla f(S_2)\| \leq L_f \|S_1 - S_2\|$, $\forall S_1, S_2$ in the domain of $f$. 
Instead of directly optimizing the cost in (P2), APG algorithms minimize a sequence of overestimators, obtained at judiciously chosen points $T$. Define $g(S) := \nu \|X\|_* + \nu \lambda \|A\|_1$ and form the quadratic approximation

$$Q(S, T) := f(T) + \langle \nabla f(T), S - T \rangle + \frac{L_f}{2} \|S - T\|_F^2 + g(S)$$

$$= \frac{L_f}{2} \|S - G\|_F^2 + g(S) + f(T) - \frac{1}{2L_f} \|\nabla f(T)\|_F^2$$

where $G := T - (1/L_f) \nabla f(T)$. With $k = 1, 2, \ldots$ denoting iterations, APG algorithms generate the sequence of iterates

$$S[k] := \arg \min_S Q(S, T[k]) = \arg \min_S \left\{ \frac{L_f}{2} \|S - G[k]\|_F^2 + g(S) \right\}$$

where the second equality follows from the fact that the last two summands in (26) do not depend on $S$. There are two key aspects to the success of APG algorithms. First, is the selection of the points $T[k]$ where the sequence of approximations $Q(S, T[k])$ are formed, since these strongly determine the algorithm’s convergence rate. The choice $T[k] = S[k] + \frac{t[k]}{\lambda \nu} (S[k] - S[k-1])$, where $t[k] = \left[ 1 + \sqrt{4t^2[k-1] + 1} \right]/2$, has been shown to significantly accelerate the algorithm resulting in convergence rate no worse than $O(1/k^2)$ [2]. The second key element stems from the possibility of efficiently solving the sequence of subproblems (27). For the particular case of (P2), note that (27) decomposes into

$$X[k + 1] := \arg \min_X \left\{ \frac{L_f}{2} \|X - G_X[k]\|_F^2 + \nu \|X\|_* \right\}$$

$$A[k + 1] := \arg \min_A \left\{ \frac{L_f}{2} \|A - G_A[k]\|_F^2 + \nu \lambda \|A\|_1 \right\}$$

where $G[k] = [G_X[k] G_A[k]]'$. Letting $S_r(M)$ with $(i, j)$-th entry given by $\text{sign}(m_{i,j}) \max\{|m_{i,j}| - \tau, 0\}$ denote the soft-thresholding operator, and $U \Sigma V' = \text{svd}(G_X[k])$ the singular value decomposition of matrix $G_X[k]$, it follows that (see, e.g. [24])

$$X[k + 1] = US \tau_f \Sigma V', \quad A[k + 1] = S \tau_f G_A[k].$$

A continuation technique is employed to speed-up convergence of the APG algorithm. The penalty parameter $\nu$ is initialized with a large value $\nu_0$, and is decreased geometrically until it reaches the target value of $\tilde{\nu}$. The APG algorithm is tabulated as Algorithm 1. Similar to [24] and [36], the iterations terminate whenever the norm of

$$Z[k + 1] := \begin{bmatrix} L_f(T_X[k] - X[k + 1]) + (X[k + 1] + RA[k + 1] - T_X[k] - RT_A[k]) \\ L_f(T_A[k] - A[k + 1]) + R'(X[k + 1] + RA[k + 1] - T_X[k] - RT_A[k]) \end{bmatrix}$$
Algorithm 1: APG solver for (P1)

\begin{enumerate}
\item \textbf{input} Y, R, λ, ν, ν₀, \bar{ν}L_f = \lambda_{\text{max}}(I_L R′|I_L R)
\item \textbf{initialize} X[0] = X[−1] = 0_{L×T}, A[0] = A[−1] = 0_{F×T}, t[0] = t[−1] = 1, and set k = 0.
\item \textbf{while} not converged do
\begin{enumerate}
\item \textbf{do}
\begin{enumerate}
\item T_X[k] = X[k] + \frac{τ[k]}{τ[k] + 1} (X[k] − X[k − 1]).
\item T_A[k] = A[k] + \frac{τ[k]}{τ[k] + 1} (A[k] − A[k − 1]).
\item G_X[k] = T_X[k] + \frac{1}{L_f} (Y − T_X[k] − RT_A[k]).
\item G_A[k] = T_A[k] + \frac{1}{L_f} R′ (Y − T_X[k] − RT_A[k]).
\end{enumerate}
\item \textbf{end do}
\begin{enumerate}
\item \textbf{end while}
\end{enumerate}
\end{enumerate}
\item \textbf{return} X[k], A[k]
\end{enumerate}

Before concluding this section, it is worth noting that Algorithm 1 has good convergence performance, and quantifiable iteration complexity as asserted in the following proposition adapted from [2], [24].

**Proposition 2** [24] Let h(·) and \{A, X\} denote, respectively, the cost and an optimal solution of (P2) when ν := \bar{ν}. For k > k₀ := \log(ν/λ) the iterates \{A[k], X[k]\} generated by Algorithm 1 satisfy

\[ |h(A[k], X[k]) − h(\bar{A}, \bar{X})| \leq \frac{4(∥A[k₀] − A∥_F^2 + ∥X[k₀] − X∥_F^2)}{(k − k₀ + 1)^2} \]

B. Alternating-direction method of multipliers (AD-MoM) algorithm

The AD-MoM is an iterative augmented Lagrangian method especially well-suited for parallel processing [4], which has been proven successful to tackle the optimization tasks encountered e.g., in statistical learning problems [27], [7]. While the AD-MoM could be directly applied to (P1), R couples the entries of A and it turns out this yields more difficult \ell_1-norm minimization subproblems per iteration. To overcome this challenge, a common technique is to introduce an auxiliary (decoupling) variable B, and formulate
the following optimization problem

\[
\begin{align*}
(P3) \quad & \min_{\{X,A,B\}} \|X\|_* + \lambda\|A\|_1 \\
& \text{s. to } Y = X + RB \\
& B = A
\end{align*}
\]

which is equivalent to (P1). To tackle (P3), associate Lagrange multipliers \( \tilde{M} \) and \( \bar{M} \) with the constraints (31) and (32), respectively. Next, introduce the quadratically \textit{augmented} Lagrangian function

\[
\mathcal{L}(X, A, B, \tilde{M}, \bar{M}) = \|X\|_* + \lambda\|A\|_1 + \langle \tilde{M}, B - A \rangle + \langle \bar{M}, Y - X - RB \rangle + \frac{c}{2}\|Y - X - RB\|_F^2 + \frac{c}{2}\|A - B\|_F^2
\]

where \( c \) is a positive penalty coefficient. Splitting the primal variables into two groups \( \{X, A\} \) and \( \{B\} \), the AD-MoM solver entails an iterative procedure comprising three steps per iteration \( k = 1, 2, \ldots \)

\[\text{[S1] Update dual variables:}\]

\[
\begin{align*}
\tilde{M}[k] &= \tilde{M}[k-1] + c(B[k] - A[k]) \\
\bar{M}[k] &= \bar{M}[k-1] + c(Y - X[k] - RB[k])
\end{align*}
\]

\[\text{[S2] Update first group of primal variables:}\]

\[
X[k+1] = \arg\min_X \left\{ \frac{c}{2}\|Y - X - RB[k]\|_F^2 - \langle \bar{M}[k], X \rangle + \|X\|_* \right\}.
\]

\[
A[k+1] = \arg\min_A \left\{ \frac{c}{2}\|A - B[k]\|_F^2 - \langle \tilde{M}[k], A \rangle + \lambda\|A\|_1 \right\}.
\]

\[\text{[S3] Update second group of primal variables:}\]

\[
B[k+1] = \arg\min_B \left\{ \frac{c}{2}\|Y - X[k+1] - RB\|_F^2 + \frac{c}{2}\|A[k+1] - B\|_F^2 - \langle R\bar{M}[k] - \tilde{M}[k], B \rangle \right\}
\]

This three-step procedure implements a block-coordinate descent on the augmented Lagrangian, with dual variable updates. The minimization (36) can be recast as (28), hence \( X[k+1] \) is iteratively updated through singular value thresholding. Likewise, (37) can be put in the form (29) and the entries of \( A[k+1] \) are updated via parallel soft-thresholding operations. Finally, (38) is a strictly convex unconstrained quadratic program, whose closed-form solution is obtained as the root of the linear equation corresponding to the first-order condition for optimality. The AD-MoM solver is tabulated under Algorithm 2. Suitable termination criteria are suggested in [7, p. 18].
Algorithm 2: AD-MoM solver for (P1)

input $Y, R, \lambda, c$

initialize $X[0] = \bar{M}[-1] = 0_{L \times T}$, $A[0] = B[0] = \tilde{M}[-1] = 0_{F \times T}$, and set $k = 0$.

while not converged do

[S1] Update dual variables:

$\tilde{M}[k] = \tilde{M}[k-1] + c(B[k] - A[k])$

$\bar{M}[k] = \bar{M}[k-1] + c(Y - X[k] - RB[k])$

[S2] Update first group of primal variables:

$U \Sigma V' = \text{svd}(Y - RA[k] + c^{-1} \bar{M}[k])$

$X[k+1] = US_{1/c}(\Sigma)V'$.

$A[k+1] = c^{-1}S_{\lambda}(\bar{M}[k] + cB[k])$.

[S3] Update second group of primal variables:

$B[k+1] = A[k+1] + (R' R + I_F)^{-1} \left[ R'(Y - X[k+1] - RA[k+1]) - c^{-1}(\tilde{M}[k] - R' \bar{M}[k]) \right]$ $k \leftarrow k + 1$

end while

return $A[k], X[k]$

Conceivably, $F$ can be quite large, thus inverting the $F \times F$ matrix $R'R + I_F$ to update $B[k+1]$ could be complex computationally. Fortunately, the inversion needs to be carried out once, and can be performed and cached off-line. In addition, to reduce the inversion cost, the SVD of the compression matrix $R = U_R \Sigma_R V'_R$ can be obtained first, and the matrix inversion lemma can be subsequently employed to obtain $[R'R + I_F]^{-1} = [I_L - CV_R CV'_R]$, where $C := \text{diag} \left( \frac{\sigma_1^2}{1+\sigma_1^2}, \ldots, \frac{\sigma_p^2}{1+\sigma_p^2} \right)$ and $p = \text{rank}(R) \ll F$. Finally, note that the AD-MoM algorithm converges to the global optimum of the convex program (P1) as stated in the next proposition.

**Proposition 3:** [4] For any value of the penalty coefficient $c > 0$, the iterates $\{X[k], A[k]\}$ converge to the optimal solution of (P1) as $k \to \infty$.

**Remark 7 (Trade-off between stability and convergence rate):** The APG algorithm exhibits a convergence rate guarantee of $O(1/k^2)$ [29], while AD-MoM only attains $O(1/k)$ [20]. For the problem considered here, APG needs an appropriate continuation technique to achieve the predicted performance [24].

Extensive numerical tests with Algorithm 1 suggest that the convergence rate can vary considerably for different choices e.g., of the matrix $R$. The AD-MoM algorithm on the other hand exhibits less variability in terms of performance, and only requires tuning $c$. It is also better suited for the constrained formulation (P1), since it does not need to resort to a relaxation.
The performance of (P1) is assessed in this section via computer simulations.

A. Exact recovery

Data matrices are generated according to $Y = X_0 + V_R A_0$. The low-rank component $X_0$ is generated from the bilinear factorization model $X_0 = WZ'$, where $W$ and $Z$ are $L \times r$ and $T \times r$ matrices with i.i.d. entries drawn from Gaussian distributions $\mathcal{N}(0, 1/L)$ and $\mathcal{N}(0, 1/T)$, respectively. Every entry of $A_0$ is randomly drawn from the set $\{-1, 0, 1\}$ with $\Pr(a_{i,j} = -1) = \Pr(a_{i,j} = 1) = \pi/2$. The columns of $V_R \in \mathbb{R}^{F \times L}$ comprise the right singular vectors of the random matrix $R = U_R \Sigma_R V_R'$, with i.i.d. Bernoulli entries with parameter $1/2$ (cf. Remark 2). The dimensions are $L = 105$, $F = 210$, and $T = 420$. To demonstrate that (P1) is capable of recovering the exact values of $\{X_0, A_0\}$, the optimization problem is solved for a wide range of values of $r$ and $s$ using the APG algorithm (cf. Algorithm 1).

Let $\hat{A}$ denote the solution of (P1) for a suitable value of $\lambda$. Fig. 1 depicts the relative error in recovering $A_0$, namely $\|\hat{A} - A_0\|_F/\|A_0\|_F$ for various values of $r$ and $s$. It is apparent that (P1) succeeds in recovering $A_0$ for sufficiently sparse $A_0$ and low-rank $X_0$ from the observed data $Y$. Interestingly, in cases such as $s = 0.1 \times FT$ or $r = 0.3 \times \min(L, T)$ there is hope for recovery. In this example, one can exactly recover $\{X_0, A_0\}$ when $s = 0.0127 \times FT$ and $r = 0.2381 \times \min(L, T)$. A similar trend is observed for the recovery of $X_0$, and the corresponding plot is omitted to avoid unnecessary repetition. For different sizes of the matrix $R$, performance results averaged over ten realizations of the experiment are listed in
Recovery performance by varying the size of $\mathbf{R}$ when $r = 10$ and $\pi = 0.05$.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$\text{rank}(\mathbf{X}_0)$</th>
<th>$|\mathbf{A}_0|_0$</th>
<th>$\text{rank}(\hat{\mathbf{X}})$</th>
<th>$|\hat{\mathbf{A}}|_0$</th>
<th>$|\hat{\mathbf{A}} - \mathbf{A}_0|_F / |\mathbf{A}_0|_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>10</td>
<td>4410</td>
<td>10</td>
<td>4419</td>
<td>$2.0809 \times 10^{-6}$</td>
</tr>
<tr>
<td>$F/2$</td>
<td>10</td>
<td>4410</td>
<td>10</td>
<td>4407</td>
<td>$6.4085 \times 10^{-5}$</td>
</tr>
<tr>
<td>$F/3$</td>
<td>10</td>
<td>4410</td>
<td>10</td>
<td>9365</td>
<td>$7.76 \times 10^{-2}$</td>
</tr>
<tr>
<td>$F/5$</td>
<td>10</td>
<td>4410</td>
<td>14</td>
<td>14690</td>
<td>$6.331 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Performance comparison of LS-PCP and Algorithm 1 averaged over ten random realizations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$r = 5$, $\pi = 0.01$</th>
<th>$r = 5$, $\pi = 0.05$</th>
<th>$r = 10$, $\pi = 0.01$</th>
<th>$r = 10$, $\pi = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS-PCP</td>
<td>0.6901</td>
<td>0.6975</td>
<td>0.7001</td>
<td>0.7023</td>
</tr>
<tr>
<td>Algorithm 1</td>
<td>$7.81 \times 10^{-6}$</td>
<td>$3.037 \times 10^{-5}$</td>
<td>$1.69 \times 10^{-5}$</td>
<td>$6.4 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table I. The smaller the compression ratio $L/F$ becomes, less observations are available and performance degrades accordingly. In particular, the error performance degrades significantly for a challenging instance where $L/F = 0.2$ and $r = 0.4 \times \min(L, F)$ (cf. the last row of Table I).

The results of [10] and [14] assert that exact recovery of $\{\mathbf{X}_0, \mathbf{A}_0\}$ from the observations $\mathbf{Y} = \mathbf{X}_0 + \mathbf{A}_0$ is possible under some technical conditions. Even though the algorithms therein are not directly applicable here due to the presence of $\mathbf{R}$, one may still consider applying PCP after suitable pre-processing of $\mathbf{Y}$. One possible approach is to find the LS estimate of the superposition $\mathbf{X}_0 + \mathbf{A}_0$ as $\hat{\mathbf{Y}} = \mathbf{R}^\dagger \mathbf{Y}$, and then feed a PCP algorithm with $\hat{\mathbf{Y}}$ to obtain $\{\mathbf{X}_0, \mathbf{A}_0\}$. Comparisons between (P1) and the aforesaid two-step procedure are summarized in Table II. It is apparent that the heuristic performs very poorly, which is mainly due to the null space of matrix $\mathbf{R}$ (when $F = 2L$) that renders LS estimation inaccurate.

**B. Unveiling network anomalies via sparsity and low rank**

In the backbone of large-scale networks, origin-to-destination (OD) traffic flows experience abrupt changes which can result in congestion, and limit the quality of service provisioning of the end users. These so-called traffic volume anomalies could be due to external sources such as network failures, denial of service attacks, or, intruders which hijack the network services [35], [23], [39]. Unveiling such anomalies is
a crucial task towards engineering network traffic. This is a challenging task however, since the available data are usually high-dimensional noisy link-load measurements, which comprise the superposition of unobservable OD flows as explained next.

Consider a backbone network with topology represented by the directed graph \( G(\mathcal{N}, \mathcal{L}) \), where \( \mathcal{L} \) and \( \mathcal{N} \) denote the set of links and nodes (routers) of cardinality \( |\mathcal{L}| = L \) and \( |\mathcal{N}| = N \), respectively. The network transports \( F \) end-to-end flows associated with specific OD pairs. For backbone networks, the number of network layer flows is typically much larger than the number of physical links \((F \gg L)\). Single-path routing is considered here to send the traffic flow from a source to its intended destination. Accordingly, for a particular flow multiple links connecting the corresponding OD pair are chosen to carry the traffic. Sparing details that can be found in [25], the traffic \( Y := [y_{l,t}] \in \mathbb{R}^{L \times T} \) carried over links \( l \in \mathcal{L} \) and measured at time instants \( t \in [1, T] \), can be compactly expressed as

\[
Y = R(Z + A) + E
\]

(39)

where the fat routing matrix \( R := [r_{\ell,f}] \in \{0, 1\}^{L \times F} \) is fixed and given, \( Z := [z_{f,t}] \) denotes the unknown ‘clean’ traffic flows over the time horizon of interest, \( A := [a_{f,t}] \) collects the traffic volume anomalies across flows and time, and \( E := [e_{l,t}] \) captures measurement errors.

Common temporal patterns among the traffic flows in addition to their periodic behavior, render most rows (respectively columns) of \( Z \) linearly dependent, and thus \( Z \) typically has low rank [23], [32]. Anomalies are expected to occur sporadically over time, and only last for short periods relative to the (possibly long) measurement interval \([1, T]\). In addition, only a small fraction of the flows are supposed to be anomalous at any given time instant. This renders the anomaly matrix \( A \) sparse across rows and columns. Given link measurements \( Y \) and the routing matrix \( R \), the goal is to estimate \( A \) by capitalizing on the sparsity of \( A \) and the low-rank property of \( Z \). Since the primary goal is to recover \( A \), define \( X := RZ \) which inherits the low-rank property from \( Z \), and consider

\[
Y = X + RA + E
\]

(40)

which is identical to (1) modulo small measurement errors in \( E \in \mathbb{R}^{L \times T} \). If \( E = 0_{L \times T} \), then (P1) can be used to unveil network anomalies, whereas (P2) is more suitable for a noisy setting.

**Remark 8 (Distributed algorithms):** Implementing Algorithms 1 and 2 presumes that network nodes communicate their local link traffic measurements to a central processing unit, which uses their aggregation in \( Y \) to determine network anomalies. Collecting all this information can be challenging due to excessive protocol overhead, or, may be even impossible in e.g., wireless sensor networks operating under stringent power budget constraints. Performing the optimization in a centralized fashion raises robustness concerns
Fig. 2. Network topology graph.

Fig. 3. Performance for synthetic data. (a) ROC curves of the proposed versus the PCA-based method with $\pi = 0.001$, $r = 10$ and $\sigma = 0.1$. (b) Amplitude of the true and estimated anomalies for $P_E = 10^{-4}$ and $P_D = 0.97$. Lines with open and filled circle markers denote the true and estimated anomalies, respectively.

as well, since the central node carrying out the specific task at hand represents an isolated point of failure. These reasons motivate devising *fully-distributed* algorithms for unveiling anomalies in large scale networks, whereby each node carries out simple computational tasks locally, relying only on its local measurements and messages exchanged with its directly connected neighbors. This is the subject dealt with in an algorithmic companion paper [26], which puts forth a general framework for in-network sparsity-regularized rank minimization.

**Synthetic network data.** A network of $N = 20$ agents is considered as a realization of the random
geometric graph model, that is, agents are randomly placed on the unit square and two agents communicate with each other if their Euclidean distance is less than a prescribed communication range of 0.35; see Fig. 2. The network graph is bidirectional and comprises \( L = 106 \) links, and \( F = N(N - 1) = 380 \) OD flows. For each candidate OD pair, minimum hop count routing is considered to form the routing matrix \( R \). With \( r = 10 \), matrices \( \{X_0, A_0\} \) are generated as explained in Section VII-A. With reference to (39), the entries of \( E \) are i.i.d., zero-mean, Gaussian with variance \( \sigma^2 \), i.e., \( e_{l,t} \sim \mathcal{N}(0, \sigma^2) \).

**Real network data.** Real data including OD flow traffic levels are collected from the operation of the Internet2 network (Internet backbone network across USA) [1]. OD flow traffic levels are recorded for a three-week operation of Internet2 during Dec. 8–28, 2008 [23]. Internet2 comprises \( N = 11 \) nodes, \( L = 41 \) links, and \( F = 121 \) flows. Given the OD flow traffic measurements, the link loads in \( Y \) are obtained through multiplication with the Internet2 routing matrix [1]. Even though \( Y \) is ‘constructed’ here from flow measurements, link loads can be typically acquired from simple network management protocol (SNMP) traces [35]. The available OD flows are a superposition of ‘clean’ and anomalous traffic, i.e., the sum of unknown ‘ground-truth’ low-rank and sparse matrices \( X_0 + A_0 \) adhering to (39) when \( R = I_L \). Therefore, PCP is applied first to obtain an estimate of the ‘ground-truth’ \( \{X_0, A_0\} \). The estimated \( X_0 \) exhibits three dominant singular values, confirming the low-rank property of \( X_0 \).

**Comparison with the PCA-based method.** To highlight the merits of the proposed anomaly detection algorithm, its performance is compared with the workhorse PCA-based approach of [23]. The crux of this method is that the anomaly-free data is expected to be low-rank, whereas the presence of anomalies considerably increases the rank of \( Y \). PCA requires a priori knowledge of the rank of the anomaly-free traffic matrix, and is unable to identify anomalous flows, i.e., the scope of [23] is limited to a single anomalous flow per time slot. Different from [23], the developed framework here enables identifying multiple anomalous flows per time instant. To assess performance, the detection rate will be used as figure of merit, which measures the algorithm’s success in identifying anomalies across both flows and time.

For the synthetic data case, ROC curves are depicted in Fig. 3 (a), for different values of the rank required to run the PCA-based method. It is apparent that the proposed scheme detects accurately the anomalies, even at low false alarm rates. For the particular case of \( P_F = 10^{-4} \) and \( P_D = 0.97 \), Fig. 3 (b) illustrates the magnitude of the true and estimated anomalies across flows and time. Similar results are depicted for the Internet2 data in Fig. 4, where it is also apparent that the proposed method markedly outperforms PCA in terms of detection performance. For an instance of \( P_F = 0.04 \) and \( P_D = 0.93 \), Fig. 4 (b) shows the effectiveness of the proposed algorithm in terms of unveiling the anomalous flows and time instants.
Fig. 4. Performance for Internet2 network data. (a) ROC curves of the proposed versus the PCA-based method. (b) Amplitude of the true and estimated anomalies for $P_F = 0.04$ and $P_D = 0.93$. Lines with open and filled circle markers denote the true and estimated anomalies, respectively.

VIII. CLOSING COMMENTS

This paper deals with recovery of low-rank plus compressed sparse matrices via convex optimization. The corresponding task arises with network traffic monitoring, brain activity detection from undersampled fMRI, and video surveillance tasks, while it encompasses compressive sampling and principal components pursuit. To estimate the unknowns, a convex optimization program is formulated that minimizes a trade-off between the nuclear and $\ell_1$-norm of the low-rank and sparse components, respectively, subject to a data modeling constraint. A deterministic approach is adopted to characterize local identifiability and sufficient conditions for exact recovery via the aforementioned convex program. Intuitively, the obtained conditions require: i) incoherent, sufficiently low-rank and sparse components; and ii) a compression matrix that behaves like an isometry when operating on sparse vectors. Because these conditions are in general NP-hard to check, it is shown that matrices drawn from certain random ensembles can be recovered with high probability. First-order iterative algorithms are developed to solve the nonsmooth optimization problem, which converge to the globally optimal solution with quantifiable complexity. Numerical tests with synthetic and real network data corroborate the effectiveness of the novel approach in unveiling traffic anomalies across flows and time.

One can envision several extensions to this work, which provide new and challenging directions for future research. For instance, it seems that the requirement of an orthonormal compression matrix is only a restriction imposed by the method of proof utilized here. There should be room for tightening the bounds
used in the process of constructing the dual certificate, and hence obtain milder conditions for exact recovery. It would also be interesting to study stability of the proposed estimator in the presence of noise and missing data. In addition, one is naturally tempted to search for a broader class of matrices satisfying the exact recovery conditions, including e.g., non block-diagonal and binary routing (compression) matrices arising with the network anomaly detection task.

APPENDIX

A. Proof of Lemma 2: Suppose \{X_0, A_0\} is an optimal solution of (P1). For the nuclear norm and the \ell_1\text{-norm at point }\{X_0, A_0\} pick the subgradients \(UV' + W_0\) and \(\text{sign}(A_0) + F_0\), respectively, satisfying the optimality condition

\[
\lambda \text{sign}(A_0) + \lambda F = R'(UV' + W).
\]

(41)

Consider a feasible solution \(\{X_0 + RH, A_0 - H\}\) for arbitrary nonzero \(H\). The subgradient inequality yields

\[
\|X_0 + RH\|_* + \lambda\|A_0 - H\| \geq \|X_0\|_* + \lambda\|A_0\|_1 + \langle UV' + W_0, RH \rangle - \lambda\langle \text{sign}(A_0) + F_0, H \rangle.
\]

(42)

To guarantee uniqueness, \(\varphi(H)\) must be positive. Rearranging terms one obtains

\[
\varphi(H) = \langle W_0, RH \rangle - \lambda\langle F_0, H \rangle + \langle R'UV' - \lambda\text{sign}(A_0), H \rangle.
\]

(43)

The value of \(W_0\) can be chosen such that \(\langle W_0, RH \rangle = \|P_{\Phi^\perp}(RH)\|_*\). This is because, \(\|P_{\Phi^\perp}(RH)\|_* = \sup \|W\| \leq 1 \|\langle W, P_{\Phi^\perp}(RH) \rangle\|\), thus there exists a \(W\) such that \(\langle P_{\Phi^\perp}(W), RH \rangle = \|P_{\Phi^\perp}(RH)\|_*\). One can then choose \(W_0 := P_{\Phi^\perp}(W)\) since \(\|P_{\Phi^\perp}(W)\| \leq \|W\| \leq 1\) and \(P_{\Phi}(W_0) = 0_{L \times T}\). Similarly, if one selects \(F_0 := -P_{\Omega^\perp}(\text{sign}(H))\), which satisfies \(P_{\Omega}(F_0) = 0_{F \times T}\) and \(\|F_0\|_{\infty} = 1\), then \((F_0, H) = -\|P_{\Omega^\perp}(H)\|_1\). Now, using (41), equation (42) is expressed as

\[
\varphi(H) = \|P_{\Phi^\perp}(RH)\|_* + \lambda\|P_{\Omega^\perp}(H)\| + \langle \lambda F - R'W, H \rangle.
\]

From the triangle inequality \(\|\langle \lambda F - R'W, H \rangle \| \leq \lambda\|F, H\| + \|R'W, H\|\), it thus follows that

\[
\varphi(H) \geq \left(\|P_{\Phi^\perp}(RH)\|_* - \|R'W, H\|\right) + \lambda \left(\|P_{\Omega^\perp}(H)\|_1 - \|F, H\|\right).
\]

(43)

Since \(P_{\Phi^\perp}(W) = W\), it is deduced that \(\|\langle W, RH \rangle\| = \|W, P_{\Phi^\perp}(RH)\| \leq \|W\||P_{\Phi^\perp}(RH)|_*\)\).

Likewise, \(P_{\Omega^\perp}(F) = F\) yields \(\|F, H\| = \|F, P_{\Omega^\perp}(H)\| \leq \|F\|_{\infty}\|P_{\Omega^\perp}(H)\|_1\). As a result

\[
\varphi(H) \geq \langle 1 - \|W\|, \|P_{\Phi^\perp}(RH)\|_* \rangle + \lambda(1 - \|F\|_{\infty})\|P_{\Omega^\perp}(H)\|_1
\]

\[
\geq (1 - \max\{\|W\|, \|F\|_{\infty}\})\|P_{\Phi^\perp}(RH)\|_* + \lambda\|P_{\Omega^\perp}(H)\|_1.
\]

(44)
Now, if \( \|W\| < 1 \) and \( \|F\|_{\infty} < 1 \), since \( \Phi \cap \Omega = \{0_{L \times T}\} \) and \( RH \neq 0_{L \times T}, \forall H \in \Omega \setminus \{0_{F \times T}\} \), there is no \( H \in \Omega \) for which \( RH \in \Phi \), and therefore, \( \varphi(H) > 0 \).

Since \( W \) and \( F \) are related through (41), upon defining \( \Gamma := R'(UV' + W) \), which is indeed the dual variable for (P1), one can arrive at conditions C1)-C4).

**B. Proof of Lemma 3**: To establish that the rows of \( A_{\Omega} \) are linearly independent, it suffices to show that \( \|A'_\text{vec}(H)\| > 0 \), for all nonzero \( H \in \Omega \). It is then possible to

\[
\|A'_\text{vec}(H)\| = \|(I - P_V) (I - P_U) R \text{vec}(H)\| = \|(I - P_U) RH (I - P_V)\|_F
\]

\[
= \|P_{\Phi^\perp}(RH)\|_F = \|RH - P_{\Phi}(RH)\|_F
\]

\[
\geq \|RH\|_F - \|P_{\Phi}(RH)\|_F \geq \|RH\|_F (1 - \mu(\Omega, \Phi))
\]

(45)

where (a) follows from the triangle inequality, and (b) from (3). The assumption \( \delta_k(R) < 1 \) along with the fact that no column of \( H \) has more than \( k \) nonzero elements, imply that \( RH \neq 0_{L \times T}. \) Since \( \mu(\Omega, \Phi) < 1 \) by assumption, the claim follows from (45).

To arrive at the desired bound on \( \sigma_{\min}(A'_\Omega) \), recall the definition of the minimum singular value [21]

\[
\sigma_{\min}(A'_\Omega) = \min_{H \in \Omega \setminus \{0_{F \times T}\}} \frac{\|A'_\text{vec}(H)\|}{\|\text{vec}(H)\|} = \min_{H \in \Omega \setminus \{0_{F \times T}\}} \frac{\|(I - P_U)RH(I - P_V)\|_F}{\|H\|_F}
\]

\[
\geq \min_{H \in \Omega \setminus \{0_{F \times T}\}} \frac{\|RH\|_F \times \|P_{\Phi^\perp}(RH)\|_F}{\|H\|_F} \geq c^{1/2} (1 - \delta_k(R))^{1/2} \min_{z \in \Omega \setminus \{0_{L \times T}\}} \frac{\|Z - P_{\Phi}(Z)\|_F}{\|Z\|_F}
\]

\[
\geq c^{1/2} (1 - \delta_k(R))^{1/2} \left(1 - \max_{z \in \Omega \setminus \{0_{L \times T}\}} \frac{\|P_{\Phi}(Z)\|_F}{\|Z\|_F}\right)
\]

\[
\geq c^{1/2} (1 - \delta_k(R))^{1/2} (1 - \mu(\Phi, \Omega))
\]

In obtaining (c), the assumption \( \delta_k(R) < 1 \) along with the fact that no column of \( H \) has more than \( k \) nonzero elements was used to ensure that \( RH \neq 0_{L \times T}. \) In addition, (d) and (f) follow from the definitions (4) and (3), respectively, while (e) follows from the triangle inequality.

**C. Proof of Lemma 4**: Towards establishing the first bound, from the submultiplicative property of the spectral norm one obtains

\[
\|Q\| = \|A_{\Omega^\perp} A'_{\Omega} (A_{\Omega} A'_{\Omega})^{-1}\| \leq \|A_{\Omega^\perp}\| \|A'_{\Omega}\| (A_{\Omega} A'_{\Omega})^{-1}\|
\]

(46)

Next, upper bounds are derived for both factors on the right-hand side of (46). First, using the fact that
\( A'A = A'_{\Omega}A_{\Omega} + A'_{\Omega^\perp}A_{\Omega^\perp} \) one arrives at
\[
\|A_{\Omega^\perp}\|^2 = \max_{x \neq 0} \frac{x'A_{\Omega^\perp}A_{\Omega^\perp}x}{\|x\|^2} = \max_{x \neq 0} \frac{x'(A'A - A'_{\Omega}A_{\Omega})x}{\|x\|^2} \leq \max_{x \neq 0} \frac{x'A_{\Omega}Ax}{\|x\|^2} - \min_{x \neq 0} \frac{x'A_{\Omega^\perp}A_{\Omega^\perp}x}{\|x\|^2} = \|A\|^2 - \sigma_{\min}^2(A'_{\Omega}). \tag{47}
\]

Note that \( A'_{\Omega} (A_{\Omega}A'_{\Omega})^{-1} \) is the pseudo-inverse of the full row rank matrix \( A_{\Omega} \) (cf. Lemma 3), and thus \( \|A'_{\Omega} (A_{\Omega}A'_{\Omega})^{-1} \| = \sigma_{\min}^{-1}(A'_{\Omega}) \) [21]. Substituting these two bounds into (46) yields
\[
\|A_{\Omega^\perp}A'_{\Omega} (A_{\Omega}A'_{\Omega})^{-1} \| \leq \left\{ \frac{\|A\|^2}{\sigma_{\min}(A'_{\Omega})} - 1 \right\}^{1/2}. \tag{48}
\]

In addition, it holds that
\[
\|A\|^2 = \lambda_{\max} \left\{ (I - P_V) \otimes R'(I - P_U)R \right\} = \lambda_{\max} \left\{ (I - P_V) \right\} \times \lambda_{\max} \left\{ R'(I - P_U)R \right\} \overset{(a)}{=} \|R'(I - P_U)\|^2 \overset{(b)}{=} 1. \tag{49}
\]

where in (a) and (b) it was used that the rows of \( R \) are orthonormal, and the maximum singular value of a projection matrix is one. Substituting (49) and the bound of Lemma 3 into (48), leads to (4).

In order to prove the second bound, first suppose that \( \|I - A_{\Omega}A'_{\Omega}\|_{\infty,\infty} < 1 \). Then, one can write
\[
\|A_{\Omega^\perp}A'_{\Omega} (A_{\Omega}A'_{\Omega})^{-1} \|_{\infty,\infty} = \|A_{\Omega^\perp}A'_{\Omega}\|_{\infty,\infty} (A_{\Omega}A'_{\Omega})^{-1} \|_{\infty,\infty} \leq \|A_{\Omega^\perp}A'_{\Omega}\|_{\infty,\infty} (I - (I - A_{\Omega}A'_{\Omega}))^{-1} \|_{\infty,\infty} \leq \frac{\|A_{\Omega^\perp}A'_{\Omega}\|_{\infty,\infty}}{1 - \|I - A_{\Omega}A'_{\Omega}\|_{\infty,\infty}}. \tag{50}
\]

In what follows, separate upper bounds are derived for \( \|A_{\Omega^\perp}A'_{\Omega}\|_{\infty,\infty} \) and \( \|I - A_{\Omega}A'_{\Omega}\|_{\infty,\infty} \). For notational convenience introduce \( S := \text{supp}(A_0) \) (resp. \( \bar{S} \) denotes the set complement). Starting with the numerator in the right-hand side of (50)
\[
\|A_{\Omega^\perp}A'_{\Omega}\|_{\infty,\infty} = \max_i \|e'_i A_{\Omega^\perp} A'_{\Omega}\|_1 = \max_i \sum_k |\langle e'_i A_{\Omega^\perp}, e'_k A_{\Omega}\rangle| = \max_j \sum_{\ell} |\langle e'_j A, e'_\ell A\rangle| = \max_j \sum_{\ell} |\langle AA'e_j, e_\ell\rangle| = \max \left\{ \sum_{(j, \ell) \in S} |\langle R'(I - P_U)e_{j1} e'_{j2} (I - P_V), e_{\ell1} e'_{\ell2}\rangle| \right\} = \max \left\{ \sum_{(j, \ell) \in S} |\langle Re_{j1} e'_{j2} (I - P_V), e_{\ell1} e'_{\ell2}\rangle| : g_{j1, j2, \ell1, \ell2} \right\}. \tag{51}
\]
Following some manipulations, the term inside the summation can be further bounded as
\[
g(j_1,j_2,\ell_1,\ell_2) = |\langle \text{Re}_{j_2} e'_{j_2}, (I - P_U) \text{Re}_{\ell_1} e'_{\ell_2} \rangle - \langle \text{Re}_{j_1} e'_{j_1} P_V (I - P_U) \text{Re}_{\ell_1} e'_{\ell_2} \rangle|
\]
\[
= |\langle e'_{j_2} e_{\ell_2}, e'_{j_1} R'(I - P_U) \text{Re}_{\ell_1} \rangle - \langle e'_{j_2} P_V e_{\ell_2}, e'_{j_1} R'(I - P_U) \text{Re}_{\ell_1} \rangle|
\]
\[
= |e'_{j_2} R'(I - P_U) \text{Re}_{\ell_1} \mathbb{1}_{\{j_2 = \ell_2\}} - (e'_{j_2} P_V e_{\ell_2} (e'_{j_1} R'(I - P_U) \text{Re}_{\ell_1})|.
\]  
(52)

Upon defining \(x_{j_1,\ell_1} := e'_{j_1} R'(I - P_U) \text{Re}_{\ell_1}\) and \(y_{j_2,\ell_2} := (e'_{j_2} P_V e_{\ell_2})\), squaring \(g\) gives rise to
\[
g^2(j_1,j_2,\ell_1,\ell_2) = x_{j_1,\ell_1}^2 \mathbb{1}_{\{j_2 = \ell_2\}} + y_{j_2,\ell_2}^2 x_{j_1,\ell_1} + 2y_{j_2,\ell_2} x_{j_1,\ell_1} \mathbb{1}_{\{j_2 = \ell_2\}}.
\]  
(53)

Since \(y_{j_2,\ell_2} \mathbb{1}_{\{j_2 = \ell_2\}} \leq \|P_V e_{j_2}\|^2 \mathbb{1}_{\{j_2 = \ell_2\}} \geq 0\), one can ignore the third summand in (53) to arrive at
\[
g(j_1,j_2,\ell_1,\ell_2) \leq x_{j_1,\ell_1} [\mathbb{1}_{\{j_2 = \ell_2\}} + y_{j_2,\ell_2}^2]^{1/2}.
\]  
(54)

Towards bounding the scalars \(x_{j_1,\ell_1}\) and \(y_{j_2,\ell_2}\), rewrite \(x_{j_1,\ell_1} := e'_{j_1} R' \text{Re}_{\ell_1} - e'_{j_1} R' P_U \text{Re}_{\ell_1}\). If \(j_1 = \ell_1\), it holds that \(x_{j_1,\ell_1} \leq \|\text{Re}_{\ell_1}\|^2 \leq c(1 + \delta_1(R))\); otherwise,
\[
x_{j_1,\ell_1} \leq |e'_{j_1} R' \text{Re}_{\ell_1}| + |e'_{j_1} R' P_U \text{Re}_{\ell_1}| \leq c\theta_{1,1}(R) + c(1 + \delta_1(R)) \gamma_R^2(U).
\]

Moreover, \(y_{j_2,\ell_2} \leq \|P_V e_{j_2}\| \|P_V e_{\ell_2}\| \leq \gamma^2(V)\). Plugging the bounds into (54) yields
\[
g(j_1,j_2,\ell_1,\ell_2) \leq [c(1 + \delta_1(R)) \mathbb{1}_{\{j_2 = \ell_2\}} + c(1 + \delta_1(R)) \gamma_R^2(U)] \mathbb{1}_{\{j_1 \neq \ell_1\}}
\]
\[
\times \mathbb{1}_{\{j_2 = \ell_2\}} + \gamma^4(V)]^{1/2}.
\]  
(55)

Plugging (55) into (51) one arrives at
\[
\|A_\Omega A'_\Omega\|_{\infty,\infty} \leq c \sqrt{2k + s \gamma^2(V)} \theta_{1,1}(R) + c(1 + \delta_1(R)) \left[ k \gamma^2(V) + \sqrt{2k} \gamma_R^2(U) + s \gamma_R^2(U) \gamma^2(V) \right]
\]
\[
:= c\omega_{\text{max}}
\]  
(56)

after using: i) \(\mathcal{S} \cap \bar{\mathcal{S}} = \emptyset\) and consequently \(j_2 \neq \ell_2\) when \(j_1 = \ell_1\); and ii) \(\gamma(V) \leq 1\).

Moving on, consider bounding \(\|I - A_\Omega A'_\Omega\|_{\infty,\infty}\) that can be rewritten as
\[
\|I - A_\Omega A'_\Omega\|_{\infty,\infty} = \max_i \|e'_i (I - A_\Omega A'_\Omega)\|_1
\]
\[
= \max_i \left\{ |1 - \|e'_i A_\Omega\|^2| + \sum_{k \neq i} |\langle e'_i A_\Omega, e_k A_\Omega \rangle| \right\}
\]
\[
= \max_i \left\{ |1 - \|A'_j e_j\|^2| + \sum_{\ell \neq j} |\langle A'_j e_j, A'_\ell e_\ell \rangle| \right\}.
\]  
(57)
In the sequel, an upper bound is derived for (57). Let \((j_1, j_2)\) denote the element of \(S\) associated with \(j\) in (57). For the first summand inside the curly brackets in (57), consider lower bounding the norm of the \(j\)-th row of \(A\) as
\[
\|A'e_j\| = \|(I - P_U)Re_j, e_{j_2}(I - P_V)\|_F = \|P_{\Phi'}(Re_j, e_{j_2})\|_F \\
= \|Re_j, e_{j_2} - P_{\Phi}(Re_j, e_{j_2})\|_F \geq \|Re_j, e_{j_2}\| - \|P_{\Phi}(Re_j, e_{j_2})\|_F \\
\geq \|Re_j, e_{j_2}\|(1 - \mu(\Phi, \Omega_R)) \geq c^{1/2}(1 - \delta_1(R))^{1/2}(1 - \mu(\Phi, \Omega_R)).
\]
Since \(\delta_1(R) < 1\) and \(\mu(\Phi, \Omega_R) < 1\), one obtains \(|1 - \|A'e_j\|^2| \leq 1 - c(1 - \delta_1(R))(1 - \mu(\Phi, \Omega_R))^2\).

For the second summand inside the curly brackets in (57), a procedure similar to the one used for bounding \(\|A_{\Omega^*} A'_{\Omega^*}\|_{\infty, \infty}\) is pursued. First, observe that
\[
\sum_{\ell \neq j} |\langle AA' e_j, e_\ell \rangle| = \sum_{\ell \neq j} |\langle (I - P_V) \otimes R'(I - P_U)Re_j, e_\ell \rangle| \\
= \sum_{(\ell_1, \ell_2) \in S \setminus \{(j_1, j_2)\}} |\langle R'(I - P_U)Re_j, e_{j_2}(I - P_V), e_\ell, e_{\ell_2} \rangle| \\
= \sum_{(\ell_1, \ell_2) \in S \setminus \{(j_1, j_2)\}} |\langle Re_j, e_{j_2}, e_{\ell_1}, e_{\ell_2} \rangle| (I - P_U)Re_j, e_{j_2}(I - P_V)\|_F \\
= \sum_{\ell \neq j} \sum_{(j_1, j_2) \in S} |\langle A'e_j, A'e_\ell \rangle| \leq c\omega_{\max}.
\]

to deduce that, up to a summand corresponding to the index pair \((j_1, j_2)\), (58) is identical to the summation in (51). Following similar arguments to those leading to (55), one arrives at
\[
\max_{j = j_1 + j_2 \in S} \sum_{\ell \neq j} |\langle A'e_j, A'e_\ell \rangle| \leq c\omega_{\max}.
\]

Putting pieces together, (57) is bounded as
\[
\|I - A_{\Omega} A'_{\Omega}\|_{\infty, \infty} \leq 1 - c(1 - \delta_1(R))(1 - \mu(\Phi, \Omega_R))^2 + c\omega_{\max}.
\]
(Note that because of the assumption \(\omega_{\max} < (1 - \delta_1(R))(1 - \mu(\Phi, \Omega_R))^2\), \(\|I - A_{\Omega} A'_{\Omega}\|_{\infty, \infty} < 1\) as supposed at the beginning of the proof. Substituting (56) and (59) into (50) yields the desired bound. ■

**D. Proof of Lemma 6**: The proof bears some resemblance with those available for the matrix completion problem [9], and PCP [10]. However, presence of the compression matrix \(R\) gives rise to unique challenges in some stages of the proof, which necessitate special treatment. In what follows, emphasis is placed on the distinct arguments required by the setting here.

The main idea is to obtain first an upper bound on the norm of the linear operator \(\pi^{-1} P_{\Phi} RP_{\Omega} R' P_{\Phi} - P_{\Phi}\), which is then utilized to upper bound \(\mu(\Phi, \Omega_R) = \|P_{\Phi} RP_{\Omega}\|\). The former is established in the next lemma; see Appendix E for a proof.
**Lemma 10:** Suppose $S := \text{supp}(A_0)$ is drawn according to the Bernoulli model with parameter $\pi$. Let $\Lambda := \sqrt{c(1 + \delta_1(R)) \gamma_2^2(U) + \gamma_2^2(V)}$, and $n := \max\{L, F\}$. Then, there are positive numerical constants $C$ and $\tau$ such that

$$\pi^{-1}\|P\Phi R\Omega R' P\Phi - \pi P\Phi\| \leq C \sqrt{\frac{\log(LF)}{\pi} + \tau \Lambda \log(n)}$$

(60)

holds with probability higher than $1 - O \left( n^{-C\pi\Lambda^r} \right)$, provided that the right-hand side is less than one.

Building on (60), it follows that

$$\|P\Phi R\Omega R' P\Phi\| - \pi \leq \|P\Phi R\Omega R' P\Phi - \pi P\Phi\|$$

(a)

$$\leq \|P\Phi R\Omega R' P\Phi - \pi P\Phi\|$$

(b)

$$\leq C \sqrt{\pi \log(LF) + \tau \pi \Lambda \log(n)}$$

(61)

where (a) and (b) come from $\|P\Phi\| \leq 1$ and the triangle inequality, respectively. In addition,

$$\|P\Omega(R'P\Phi(X))\|_F^2 = |\langle P\Omega(R'P\Phi(X)), P\Omega(R'P\Phi(X)) \rangle|$$

$$= |\langle P\Phi(R(P\Omega(R'P\Phi(X))))), X \rangle|$$

$$\leq \|P\Phi(R(P\Omega(R'P\Phi(X))))\|_F \|X\|_F$$

(62)

for all $X \in \mathbb{R}^{L \times F}$. Recalling the definition of the operator norm, it follows from (62) that $\mu(\Phi, \Omega_R) \leq \sqrt{c^{-1}(1 - \delta_k(R))^{-1}} \|P\Phi R\Omega R' P\Phi\|^{1/2}$. Plugging the bound (61), the result follows readily. ■

**E. Proof of Lemma 10:** Start by noting that

$$R'P\Phi(X) = \sum_{i,j} \langle R'P\Phi(X), e_i e_j' \rangle e_i e_j' = \sum_{i,j} \langle X, P\Phi(Re_i e_j') \rangle e_i e_j'$$

and apply the sampling operator to obtain

$$P\Omega(R'P\Phi(X)) = \sum_{i,j} b_{i,j} \langle X, P\Phi(Re_i e_j') \rangle e_i e_j'$$

where $\{b_{i,j}\}$ are Bernoulli-distributed i.i.d. random variables with $\Pr(b_{i,j} = 1) = \pi$. Then,

$$P\Omega(RP\Omega(R'P\Phi(X))) = \sum_{i,j} b_{i,j} \langle X, P\Phi(Re_i e_j') \rangle P\Phi(Re_i e_j').$$

(63)

Moreover, since $RR' = I_L$ one finally arrives at

$$P\Phi(X) = P\Phi(RR'P\Phi(X)) = \sum_{i,j} b_{i,j} \langle X, P\Phi(Re_i e_j') \rangle P\Phi(Re_i e_j').$$

(64)
The next bound will also be useful later on

\[
\| \mathcal{P}_\Phi(\text{Re}_i e_j') \|_F^2 = \langle \mathcal{P}_\Phi(\text{Re}_i e_j'), \text{Re}_i e_j' \rangle \\
= \langle \mathcal{P}_U \text{Re}_i e_j' + \text{Re}_i e_j' P_V - \mathcal{P}_U \text{Re}_i e_j' P_V, \text{Re}_i e_j' \rangle \\
= \langle \mathcal{P}_U \text{Re}_i e_j', \text{Re}_i e_j' \rangle + \langle \text{Re}_i e_j' P_V, \text{Re}_i e_j' \rangle - \langle \mathcal{P}_U \text{Re}_i e_j' P_V, \text{Re}_i e_j' \rangle \\
^{(a)} = \| \mathcal{P}_U \text{Re}_i e_j' \|_F^2 + \| \text{Re}_i e_j' P_V \|_F^2 - \| \mathcal{P}_U \text{Re}_i e_j' \|_F^2 \| P_V e_j \|_F^2 \\
\leq c(1 + \delta_1(R)) \gamma_R^2(U) + c(1 + \delta_1(R)) \gamma^2(V) = \Lambda^2
\]  

(65)

where (a) holds because \langle \mathcal{P}_U \text{Re}_i e_j' P_V, \text{Re}_i e_j' \rangle = \langle e'_i R \mathcal{P}_U \text{Re}_i, e'_j P_V e_j \rangle and \mathcal{P}_U = \mathcal{P}_U^2 (likewise \mathcal{P}_V).

Defining the random variable \(\Xi := \pi^{-1} \| \mathcal{P}_\Phi R \mathcal{P}_U R' \mathcal{P}_\Phi - \pi \mathcal{P}_\Phi \|\) and using (64), one can write

\[
\Xi = \pi^{-1} \sup_{\| X \|_F = 1} \left\| \sum_{i,j} (b_{i,j} - \pi)(X, \mathcal{P}_\Phi(\text{Re}_i e_j')) \mathcal{P}_\Phi(\text{Re}_i e_j') \right\|_F \\
= \pi^{-1} \sup_{\| \text{vec}(X) \|_1 = 1} \left\| \sum_{i,j} (b_{i,j} - \pi) \text{vec}(X)' \text{vec}[\mathcal{P}_\Phi(\text{Re}_i e_j')] \otimes \text{vec}[\mathcal{P}_\Phi(\text{Re}_i e_j')] \right\| \\
= \pi^{-1} \left\| \sum_{i,j} (b_{i,j} - \pi) \text{vec}[\mathcal{P}_\Phi(\text{Re}_i e_j')] \otimes \text{vec}[\mathcal{P}_\Phi(\text{Re}_i e_j')] \right\|. \\
(66)
\]

Random variables \(\{b_{i,j} - \pi\}\) are i.i.d. with zero mean, and thus one can utilize the spectral concentration inequality in [33, Lemma 3.5] to find

\[
E[\Xi] \leq C \sqrt{\frac{\log(LF)}{\pi}} \max_{i,j} \| \mathcal{P}_\Phi(\text{Re}_i e_j') \|_F \leq C \sqrt{\frac{\log(LF)}{\pi}} \Lambda
\]

(67)

for some constant \(C > 0\), where (b) is due to (65). Now, applying Talagrand’s concentration tail bound [34] to the random variable \(\Xi\) yields

\[
\Pr(|\Xi - E[\Xi]| \geq t) \leq 3 \exp \left( -\frac{t \log(2)}{K} \pi \min\{1, t\} \right) \\
(68)
\]

for some constant \(K > 0\), where \(t := \tau \Lambda \log(n)\) and \(n := \max\{L, F\}\). The arguments leading to (67) and (68) are similar those used in [9, Theorem 4.2] for the matrix completion problem, and details are omitted here. Putting (67) and (68) together it is possible to infer

\[
\Xi \leq E[\Xi] + t \leq C \sqrt{\frac{\log(LF)}{\pi}} + \tau \Lambda \log(n) \\
(69)
\]

with probability higher than \(1 - O(n^{-C_\pi \Lambda^2})\), which completes the proof of the lemma. \(\blacksquare\)
REFERENCES


