Sunday Version

1 11/24 Question 1

Prove the following result: Suppose $X_1, X_2, \ldots$ is a sequence of random variables such that $X_n \xrightarrow{p} c$ where c is a constant and $n \to \infty$. If g is continuous at c then $g(X_n) \xrightarrow{p} g(c)$ as $n \to \infty$.

Answer:

Given that $g(c)$ is continous and $\epsilon > 0$, $\quad \exists\, \delta$ s.t. $|g(X_n) - g(c)| < \epsilon$ whenever $|X_n - c| < \delta$.

Thus, $\mathrm{P}(|g(X_n) - g(c)| < \epsilon) = P(|X_n - c| < \delta) \quad \forall\, \epsilon, \delta > 0$, since the initial ϵ was arbitrary positive.

Converging in probability means that:

$\mathrm{P}(|X_n - c| < \delta) = 1 \quad \forall\ \delta > 0, \quad n \to \infty.$

Substitution yields the desired result:

$\mathrm{P}(|g(X_n) - g(c)| < \epsilon = 1 \quad \forall\ \epsilon > 0, \quad n \to \infty.$

2 11/24 Question 2

Suppose that $X_1, X_2, \ldots$ are a random sample from a Poisson distribution having mean 10 (i.e., $X_1, X_2, \cdots \sim$iid Poi[10]). Prove that $e^{-\bar{x}} \xrightarrow{p} e^{-10}$.

Answer:

Using the Weak Law of Large Numbers (because we have an iid random sample), we know that $\bar{x} \xrightarrow{p} \mu$.

Thus, $\bar{x} \xrightarrow{p} 10$.

Using the result from the previous problem, a transformation of an rv converges in probability to the transformation of the convergence of the rv.

$\therefore e^{-\bar{x}} \xrightarrow{p} e^{-10}$.

3 5.1

Color blindness appears in 1% of people in a certain population. How large must a sample be if the probability of its containing a color-blind person is to be .95 or more? (Assume the population is large enough to be considered infinite, so that sampling can be considered with replacement.

Answer:

Let $X_1, X_2, \ldots$ be the random sample of the population.

Each is an iid Bernoulli where P(success)=P(color-blind)=.01.

To simplifly calculation, we will use the following fact:

P(at least 1 colorblind) = 1 - P(none colorblind)=1 - P($X_i = 0\, \forall\ i$).

$\therefore .95 > 1 - (.99)^n \Longrightarrow (.99)^n > .05.$

$n \ln .99 > \ln .05 \rightarrow n > \frac{\ln .05}{\ln .99} \approx 298.$

4 5.4

A generalization of iid random variables is exchangeable random variables...The random variables $X_1, X_2, \ldots$ are exchnageable is any permutation of any subset of them of size $k (k \leq n)$ has the same distribution... Let $X_i | P \sim$iid Bernoulli$(P), i = 1, 2, \ldots, n$ and let $P \sim$uniform$(0, 1)$.

(a) Show that marginal distriubtion of any k of the Xs iis the same as

$P(X_1 = x_1, \ldots, X_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp = \frac{t!(k-t)!}{(k+1)!}$

where $t = \sum_{i=1}^k x_i$.

Answer:

$P(X_1 = x_1, \ldots, X_k = x_k) = P(X_1 = x_1, \ldots, X_k = x_k | P) \cdot P(P).$

Since the trials are iid,

$P(X_1 = x_1, \ldots, X_k = x_k | P) = p^{x_1}(1-p)^{1-x_1} \cdot \ldots \cdot p^{x_k}(1-p)^{1-x_k} = p^t (1-p)^{k-t}.$

$P(P) = \int_0^1 dp$. We will leave it in this form for use later.

$P(X_1 = x_1, \ldots, X_k = x_k) = \int_0^1 p^t (1-p)^{k-t} dp.$

$p^t (1-p)^{k-t} dp$ is the kernal for a Beta $(t+1, k-t+1)$ distribution.

$\therefore \int_0^1 p^t (1-p)^{k-t} dp = \frac{\Gamma(t+1)\Gamma(k-t+1)}{\Gamma(t+1+k-t_1)} = \frac{t!(k-t)!}{(k+1)!}.$

(b) Show that, marginally,

$P(X_1 = x_1, \ldots, X_k = x_k) \neq \prod_{i=1}^n P(X_i = x_i)$

so the distribution of Xs is exchangeable but not iid.

Answer:

$P(X_i = x_i) = P(X_i = x_i | P) \cdot P(P) = \int_0^1 p_i^x (1-p)^{1-x_i} dp = \frac{x_i!(1-x_i)!}{(2)!}$

$\frac{t!(n-t)!}{(n+1)!} = \prod_{i=1}^n \frac{x_i!(1-x_i)!}{2!}$

5 5.16

Let X_i, $i = 1, 2, 3$ be independent with Normal(i, i^2) distributions...Use $X_i s$ to construct the following stats:

(a) χ_3^2 (b) t_2 (c) $F_{1,2}$

Answer:

(a) χ^2 are squares of standard normals, or alternately Gamma$(1/2, 1/2) \sim \chi^2_1$. Sums of squares of n standard normals are χ^2_n.

$\sum_{i=1}^{3} \left(\frac{x_i - i}{i}\right)^2 \sim \chi^2_3$.

(b) If $Z \sim N(0,1), U \sim \chi^2_n$,with Z ind U, $\frac{Z}{U/n} \sim t_n$.

$$\therefore \frac{\left(\frac{x_i - i}{i}\right)^2}{\sqrt{\frac{\sum_{j \neq i}\left(\frac{x_j - i}{j}\right)^2}{2}}} \sim t_2.$$

(c) If $U_m \sim \chi^2_m$, $V_n \sim \chi^2_n$ with U ind V, $\frac{U/m}{V/n} \sim F_{m,n}$.

$$\frac{\left(\frac{x_i - i}{i}\right)^2/1}{\sum_{j \neq i}\left(\frac{x_j - i}{j}\right)^2/2} \sim F_{1,2}$$