

**FINAL VERSION****1 2.14**

Part A: Let  $X$  be a continuous, nonnegative random variable [ $f(x) = 0$  for  $x < 0$ ]. Show that

$$EX = \int_0^{\infty} [1 - F_X(x)] dx$$

where  $F_X(x)$  is the cdf of  $X$ .

*Answer:*

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x f_X(x) dx, \text{ because } f(x) = 0 \text{ for } x < 0.$$

$$\text{Solving } \frac{d}{dx} F_X(x) = f_X(x) \text{ for } f_X(x) dx \text{ yields } d[F_X(x)] \text{ and } EX = \int_0^{\infty} x * d[F_X(x)].$$

After integrating by parts ( $u = x, du = dx, v = F_X(x), dv = dF_X(x)$ ) and inserting some convenient terms:

$$EX = xF_X(x)|_0^{\infty} - \int_0^{\infty} F_X(x) dx + x|_0^{\infty} - x|_0^{\infty}.$$

$$EX = [-x(1 - F_X(x))]|_0^{\infty} + \int_0^{\infty} (dx) - \int_0^{\infty} F_X(x) dx.$$

Which when evaluated, and simplified, yields:

$$EX = 0 - 0 + \int_0^{\infty} (dx) - F_X(x) dx = \int_0^{\infty} [1 - F_X(x)] dx \text{ the expression of interest.}$$

Part B: Let  $X$  be a discrete random variable whose range is the nonnegative integers. Show that

$$EX = \sum_0^{\infty} [1 - F_X(x)] dx$$

where  $F_X(k) = P(X \leq k)$ . Compare this with part (a).

*Answer:*

$$EX = \sum_{x \in \mathcal{X}} x * f_X(x) = \sum_0^{\infty} x * f_X(x) = \sum_0^{\infty} x * P(X = x)$$

$$EX = \sum_0^{\infty} x * (F_X(x) - F_X(x - 1)) \text{ based on the properties of cdf's.}$$

Inserting some convenient terms:

$$EX = \sum_0^{\infty} x * ((F_X(x)) - F_X(x - 1) + 1 - 1)$$

And rearranging the expression:

$$EX = \sum_0^{\infty} x * (-(1 - F_X(x)) + (1 - F_X(x - 1))) = \sum_0^{\infty} x * ((1 - F_X(x - 1)) - (1 - F_X(x)))$$

The summand term for  $x = 0$  is 0 so, splitting the summation:

$$EX = \sum_{x=1}^{\infty} x(1 - F_X(x - 1)) - \sum_{x=0}^{\infty} x(1 - F_X(x))$$

Reindexing:

$$EX = \sum_{x=0}^{\infty} (x + 1)(1 - F_X(x)) - \sum_{x=0}^{\infty} x(1 - F_X(x))$$

$$\therefore EX = \sum_0^{\infty} [1 - F_X(x)]$$

An intergral is a Summand taken over infinitely small intervals, and since  $1 - F_X(x)$  is monotonic, it is intuitively clear that the limit of the summand would be an integral, linking parts A and B.

**2 2.24**

Compute  $EX$  and  $\text{Var}X$  for each of the following probability distributions.

(a)  $f_X(x) = ax^{a-1}, 0 < x < 1, a > 0$

$$EX = \int_{-\infty}^{\infty} xax^{a-1}dx = \int_0^1 ax^a dx = \left. \frac{ax^{a+1}}{a+1} \right|_0^1 = \left( \frac{a1^{a+1}}{a+1} - \frac{a0^{a+1}}{a+1} \right) = \frac{a}{a+1}$$

$$EX^2 = \int_{-\infty}^{\infty} x^2ax^{a-1}dx = \int_0^1 ax^{a+1}dx = \left. \frac{ax^{a+2}}{a+2} \right|_0^1 = \left( \frac{a1^{a+2}}{a+2} - \frac{a0^{a+2}}{a+2} \right) = \frac{a}{a+2}$$

$$\therefore \text{Var}X = EX^2 - (EX)^2 = \frac{a}{a+2} - \frac{a^2}{(a+1)^2}$$

(b)  $f_X(x) = 1/n, x = 1, 2, \dots, n, n > 0$  an integer

$$EX = \sum_{i=1}^n i \frac{1}{n} = \frac{1}{n} + \frac{2}{n} + \dots + \frac{n}{n} = \frac{1+2+\dots+n}{n} = \frac{n(n+1)}{2n} = \frac{(n+1)}{2}.$$

$$EX^2 = \sum_{i=1}^n i^2 \frac{1}{n} = \frac{1^2}{n} + \frac{2^2}{n} + \dots + \frac{n^2}{n} = \frac{1^2+2^2+\dots+n^2}{n} = \frac{n(n+1)(n+2)}{6n} = \frac{(n+1)(n+2)}{6}.$$

$$\therefore \text{Var}X = EX^2 - (EX)^2 = \frac{(n+1)(n+2)}{6} - \frac{(n+1)^2}{2^2} = \frac{n^2-1}{12} = \frac{(n+1)(n-1)}{12}$$

(c)  $f_X(x) = \frac{3}{2}(x-1)^2, 0 < x < 2$

$$EX = \int_{-\infty}^{\infty} x(3/2)(x-1)^2 dx = (3/2) \int_0^2 x^3 - 2x^2 + x dx$$

$$EX = (3/2)(x^4/4) \Big|_0^2 - (3/2)(2x^3/3) \Big|_0^2 + (3/2)(x^2/2) \Big|_0^2 = 6 - 8 + 1 = 1.$$

$$\text{Similarly, } EX^2 = \int_{-\infty}^{\infty} x^2(3/2)(x-1)^2 dx = (3/2) \int_0^2 x^3 - 2x^2 + x dx$$

$$EX^2 = (3/2)(x^5/5) \Big|_0^2 - (3/2)(2x^4/4) \Big|_0^2 + (3/2)(x^3/3) \Big|_0^2 = 8/5.$$

$$\therefore \text{Var}X = EX^2 - (EX)^2 = 8/5 - 1^2 = 3/5$$

**3 2.33a**

Verify the expression given for the mgf and use the mgf to calculate  $EX$  and  $\text{Var}X$ .

$$(a) P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, M_X(t) = e^{\lambda(e^t-1)}, x = 0, 1, \dots; \lambda > 0.$$

*Answer:*

$$\text{Fact: For any } \lambda \in \mathcal{R}, \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^\lambda.$$

$$M_X(t) = \mathbb{E}e^{tx} = \sum_{x \in \mathcal{X}} g(x) * P(X = x) = \sum_0^\infty e^{tx} * \frac{e^{-\lambda}\lambda^x}{x!}.$$

$$M_X(t) = e^{-\lambda} \sum_0^\infty \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} * e^{-\lambda e^t} = e^{\lambda(e^t-1)}. t \in (-\infty, \infty) \text{ so mgf exists.}$$

$$EX^1 = M_X^1(0) = \frac{d^1}{dt^1} M_X(t)|_{t=0}$$

$$e^{\lambda(e^t-1)} * \lambda * e^t|_{t=0} = e^{\lambda(1-1)} * \lambda * e^0 = \lambda.$$

$$EX^2 = M_X^2(0) = \frac{d^2}{dt^2} M_X(t)|_{t=0}$$

$$EX^2 = \lambda * e^{\lambda(e^t-1)+t} \frac{d}{dt}|_{t=0} = \lambda \left( e^{\lambda(e^t-1)+t} * (\lambda(e^t - 1) + t \frac{d}{dt}) \right) |_{t=0}.$$

$$EX^2 = \lambda \left( e^{\lambda(e^t-1)+t} * (\lambda e^t + 1) \right) |_{t=0}.$$

$$EX^2 = \lambda(\lambda + 1) = \lambda^2 + \lambda.$$

$$\therefore \text{Var}X = EX^2 - (EX)^2 = (\lambda^2 + \lambda) - (\lambda^2) = \lambda.$$

## 4 2.34

A distribution cannot be uniquely defined by a finite collection of moments... Let  $X$  have the normal distribution, that is  $X$  has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

Define a discrete rv  $Y$  by

$$P(Y = \sqrt{3}) = P(Y = -\sqrt{3}) = 1/6, \quad P(Y = 0) = 2/3.$$

Show that

$$EX^r = EY^r \text{ for } r = 1, 2, 3, 4, 5.$$

*Answer:*

$$M_X(t) = Ee^{tx} = \int_{-\infty}^{\infty} e^{tx} * \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx-x^2/2} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{2tx-x^2}{2}}.$$

Which, after inserting some convenient terms and rearranging becomes:

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2+2tx-t^2+t^2}{2}} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x^2-2tx+t^2)+t^2}{2}}.$$

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-t)^2+t^2}{2}} = e^{t^2/2} * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-t)^2}{2}}.$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-t)^2}{2}} = 1 \text{ because it is the linear transformation of a Normal Distribution.}$$

$$\therefore M_X(t) = e^{t^2/2}.$$

$$M_Y(t) = EY^r = \sum_{y \in \mathcal{Y}} e^{ty} * P(Y = y), \quad \mathcal{Y} = \{-\sqrt{3}, 0, \sqrt{3}\}.$$

$$M_Y(t) = \frac{1}{6} e^{t\sqrt{3}} + \frac{1}{6} e^{t\sqrt{-3}} + \frac{2}{3} e^{t*0}$$

$$M_Y(t) = \frac{1}{6} (e^{t\sqrt{3}-t\sqrt{-3}}) + \frac{2}{3}$$

The following table shows first 5 derivatives of the the MGF's for  $X$  and  $Y$  and what they evaluate to:

r	$M_X^r(t)$	$M_Y^r(t)$	$M_X^r(0)$	$M_Y^r(0)$
1	$te^{t^2/2}$	$\frac{1}{6}[\sqrt{3}^1 e^{t\sqrt{3}} + (-\sqrt{3})^1 e^{-t\sqrt{3}}]$	0	0
2	$t^2 e^{t^2/2} + e^{t^2/2}$	$\frac{1}{6}[\sqrt{3}^2 e^{t\sqrt{3}} + (-\sqrt{3})^2 e^{-t\sqrt{3}}]$	1	1
3	$t^3 e^{t^2/2} + 3te^{t^2/2}$	$\frac{1}{6}[\sqrt{3}^3 e^{t\sqrt{3}} + (-\sqrt{3})^3 e^{-t\sqrt{3}}]$	0	0
4	$t^4 e^{t^2/2} + 6t^2 e^{t^2/2} + 3e^{t^2/2}$	$\frac{1}{6}[\sqrt{3}^4 e^{t\sqrt{3}} + (-\sqrt{3})^4 e^{-t\sqrt{3}}]$	3	3
5	$t^5 e^{t^2/2} + 10t^3 e^{t^2/2} + 15te^{t^2/2}$	$\frac{1}{6}[\sqrt{3}^5 e^{t\sqrt{3}} + (-\sqrt{3})^5 e^{-t\sqrt{3}}]$	0	0

## 5 3.5

A standard drug is known to be effective in 80% of cases in which it is used. A new drug is tested on 100 patients and found to be effective in 85 cases. Is the new drug superior? (Hint: Evaluate the probability of observing 85 or more successes assuming that the new and old drugs are equally effective.)

*Answer:*

Let  $S$  be a drug success.

$$P(S \geq 85) = \sum_{i=85}^{100} \binom{100}{i} .8^i * .2^{100-i} = 0.1285.$$

While promising this is not enough to choose the new drug over the old. A larger sample size with the same proportion would be more convincing.

For example, if there were 850 successes out of 1000 patients, the probability drops to 2.64e05.