Generalized mixed equitable partitioning

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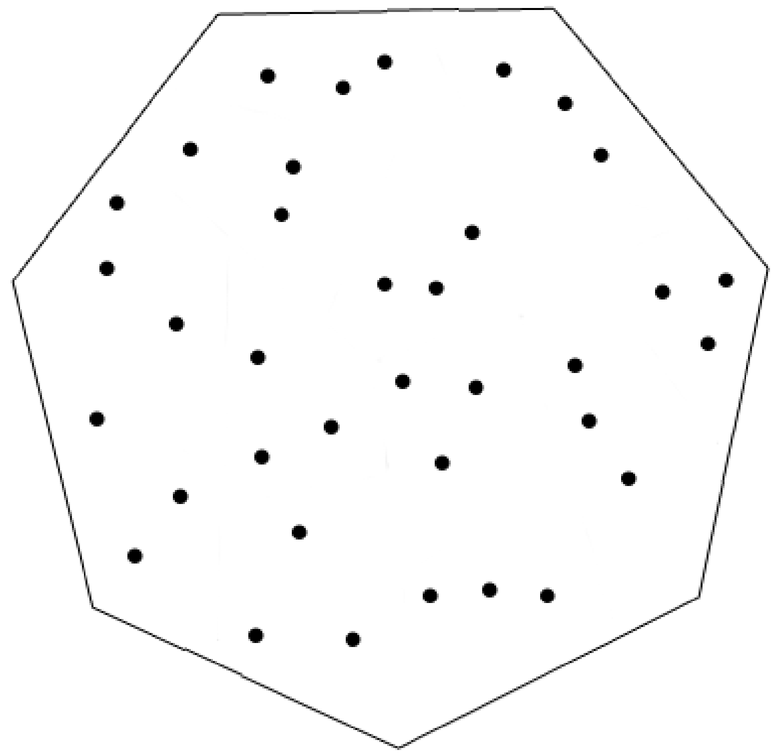
The problem

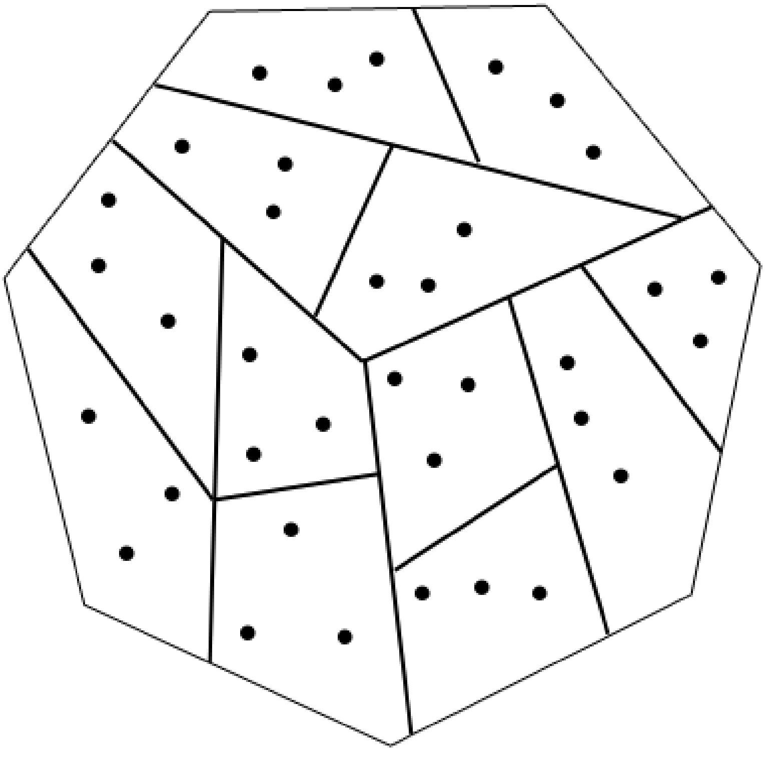
We are given a convex polygon C containing gn points. Can we break C into g subregions, satisfying:

- Each region is convex
- Each region contains n points
- Each region has equal area

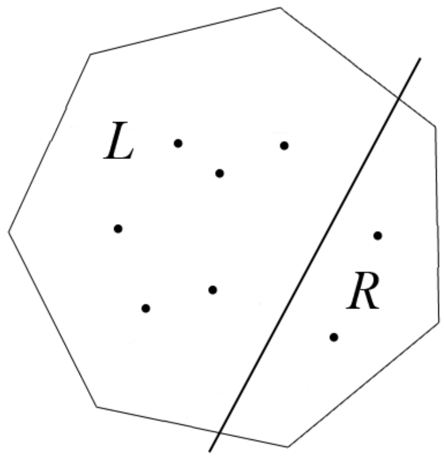


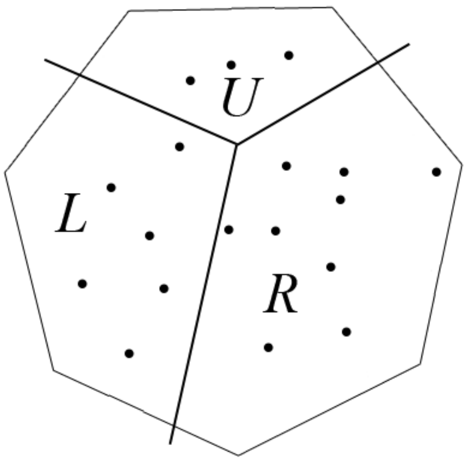
Yes

- We can find such a solution, most likely with ~cubic running time
- (our previous algorithm solves the special case $n = 1$)



Definition: Convex Equitable 2- and 3-Partitions





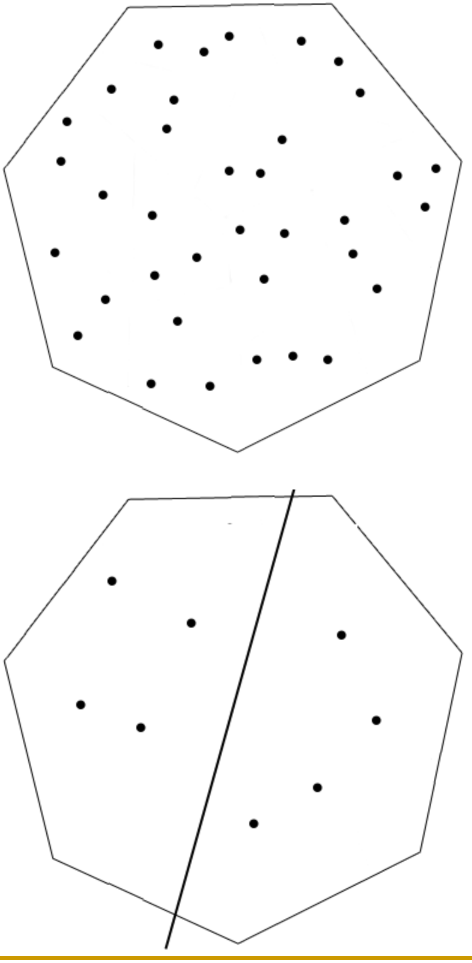
$$\frac{\text{Points in } L}{\text{Area of } L} = \frac{\text{Points in } R}{\text{Area of } R}$$

$$\frac{\text{Points in } L}{\text{Area of } L} = \frac{\text{Points in } R}{\text{Area of } R} = \frac{\text{Points in } U}{\text{Area of } U}$$

Claim: A convex equitable 2- or 3-partition always exists; we then perform this recursively

Some comments

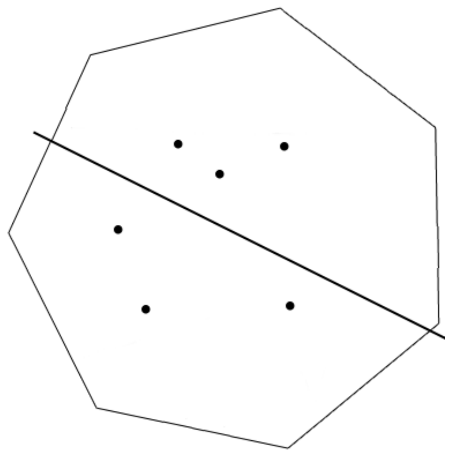
- As before, our algorithm builds recursive equitable 2- and 3-partitions
- If g is even, a ham sandwich cut is a valid 2-partition (divides area and point set in half)



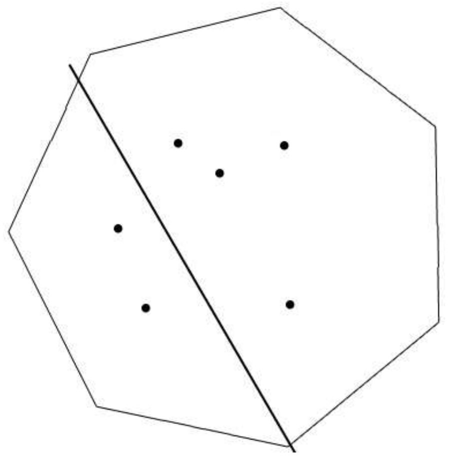
Comment

- Not all equitable partitions are acceptable! ($n = 2$, $g = 3$)

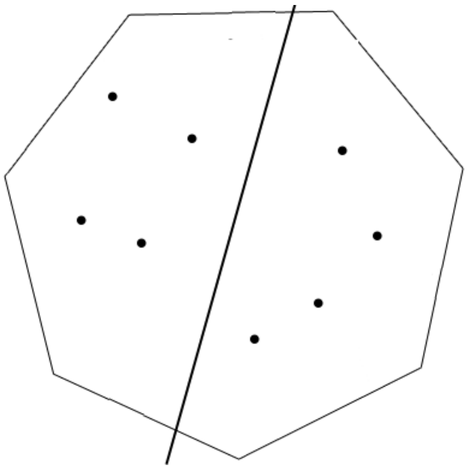
NO GOOD!



GOOD!



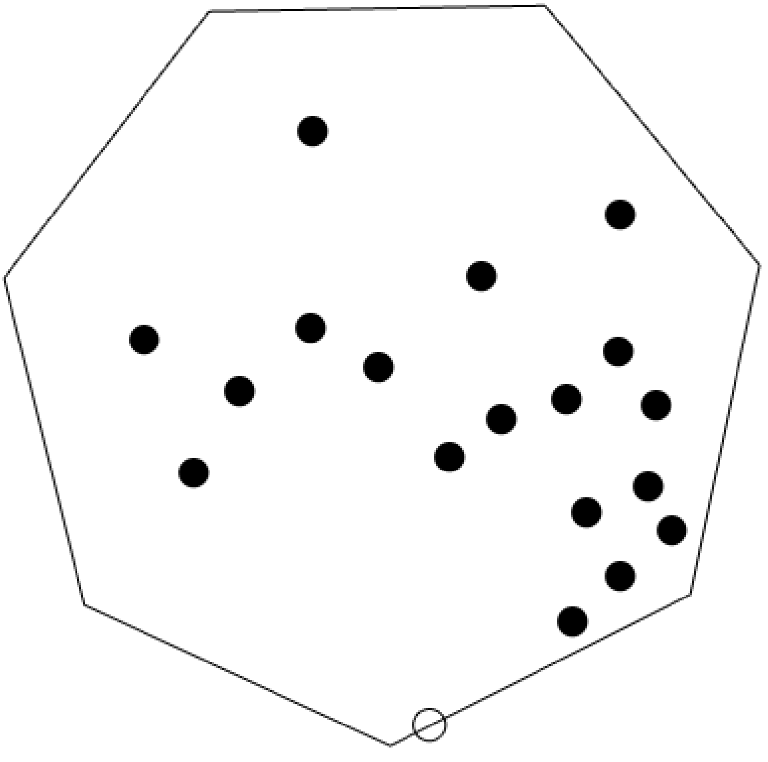
Step 0



- If g is even, build a ham-sandwich cut; we're done

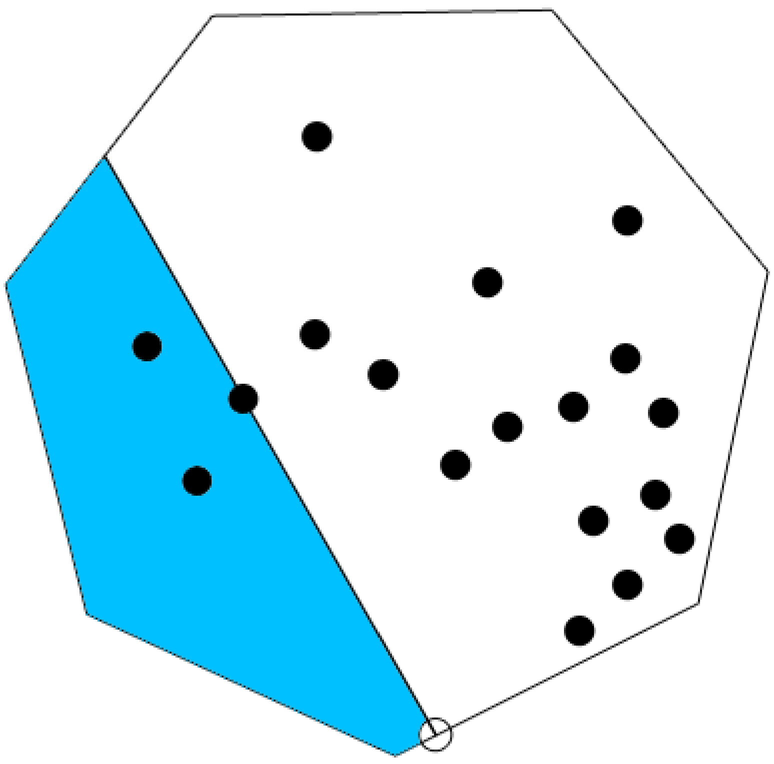
Step 1

- Pick an arbitrary point (x_0,y_0) on the boundary

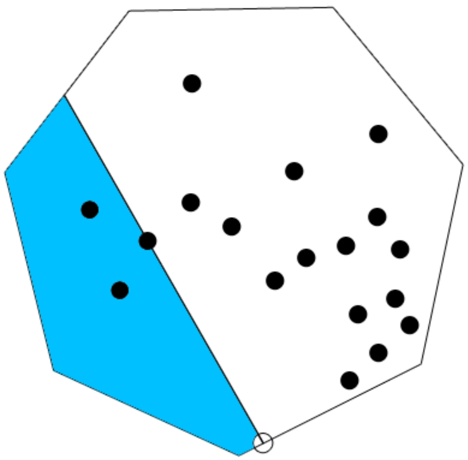


Step 1

- Pick an arbitrary point (x_0,y_0) on the boundary
- Check all half-spaces through (x_0,y_0) that cut off n, $2n$, $3n$,…,$(g\text{-}1)n$ points to the left



Step 1



Points cut off	n	$2n$	$3n$	...	$(g\text{-}1)n$
Size?	+	+	-		-

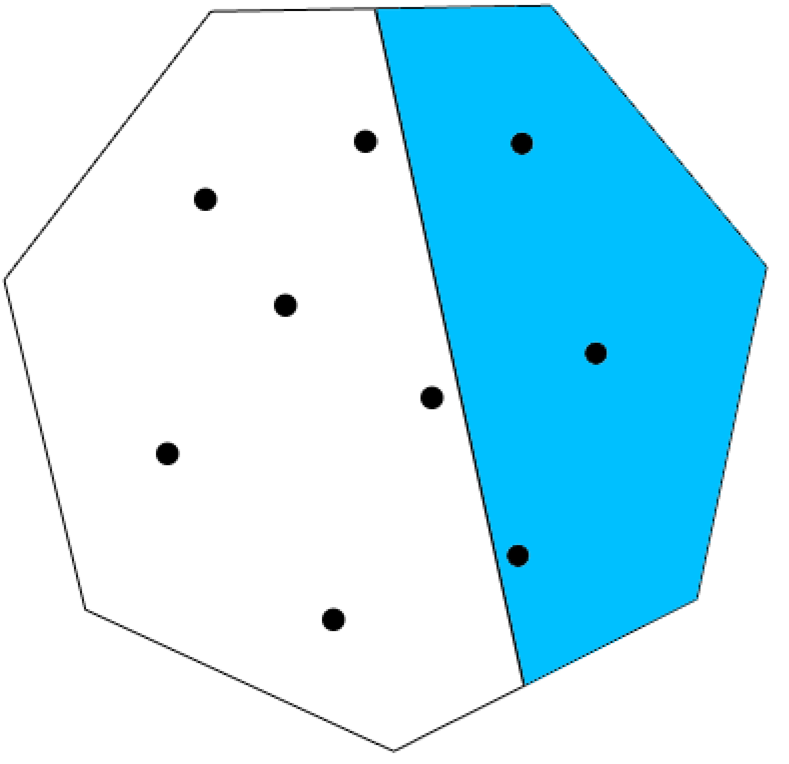
- Check the sizes of these half-spaces; if an equitable half-space exists, we're done
- Otherwise, the half-spaces through (x_0, y_0) are either too big (+) or too small (-)

Combinatorial lemma

- For integers $1,\ldots,g$-1, we have a label + or –
- There exist integers g_1, g_2, g_3 such that
  - $g_1 + g_2 + g_3 = g$
  - g_1, g_2, g_3 have the same label
- Next, we will look at *all* half-spaces cutting off g_1, g_2, or g_3 points

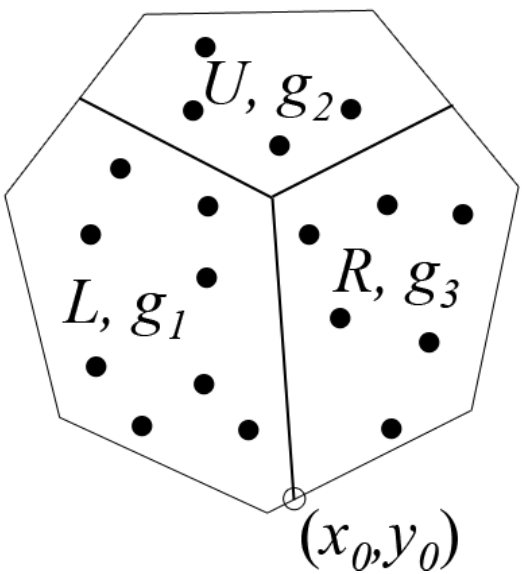
Step 2

- Get g_1, g_2, g_3 from the previous slide. Check *all* half-spaces cutting off g_1, g_2, or g_3 points. Either:
  1. All half-spaces cutting off g_1, g_2, or g_3 have the same sign, or
  2. We will have found an equitable 2-partition of the form $(g_1, g\text{-}g_1)$, $(g_2, g\text{-}g_2)$, or $(g_3, g\text{-}g_3)$ and we're done

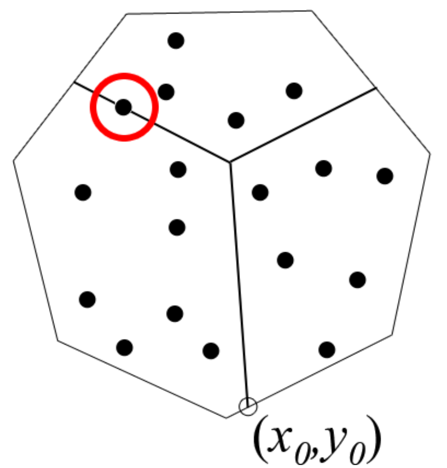


Topological lemma

- Suppose we have case 1, so all half-spaces cutting off g_1, g_2, or g_3 have the same sign (too large, for example). Then there exists a convex equitable 3-partition through (x_0, y_0) into (g_1, g_2, g_3).



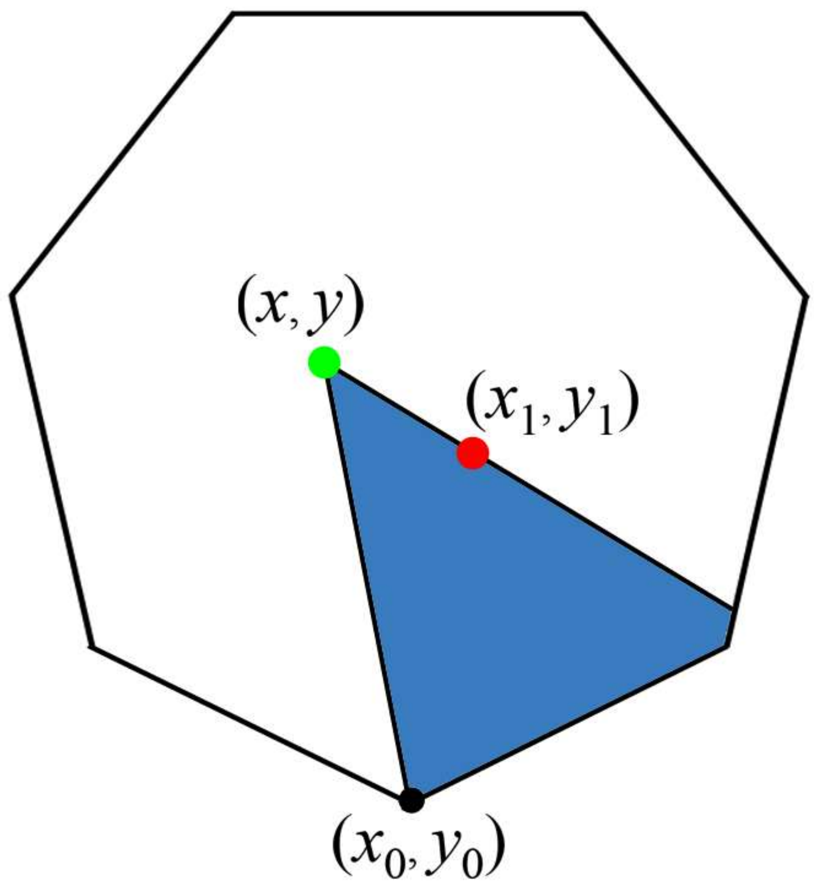
Geometric Lemma



- We can assume WLOG that one of the subregion boundaries intersects the point set
- Thus, we only have to consider the 3-partitions that intersect the point set

Geometry question

- Choose (x_0, y_0) on the boundary of C; choose points (x_1, y_1) and (x, y) in the interior of C. Consider the area spanned by sector (x_0,y_0)- (x,y)- (x_1,y_1). This region will be part R of a 3-partition. We want it to have area (g_3/g) and contain g_3 points.
- Can we describe all values (x,y) that give this "sector region" the correct area?



Observation

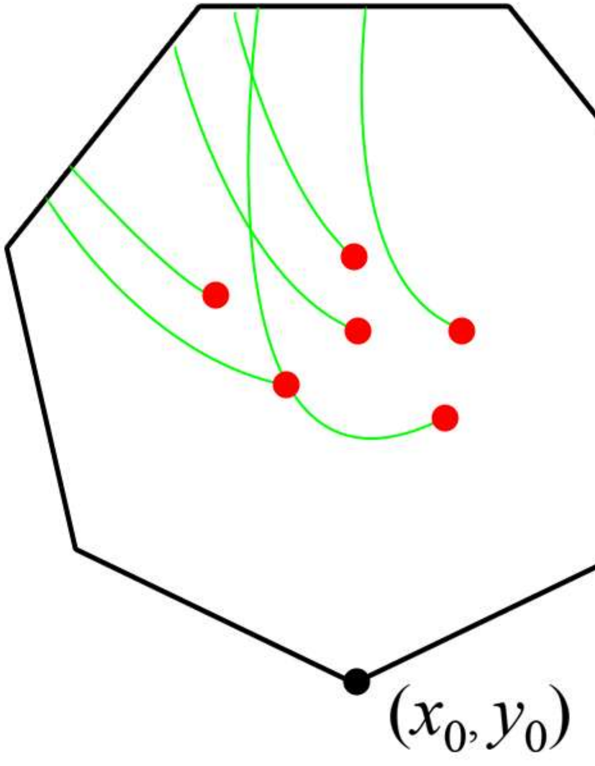
- The function measuring the area of the sector is piecewise linear-fractional in (x, y), and therefore its level sets consist of piecewise hyperbolic arcs

Observation

- Thus for each element of the point set, we have a hyperbolic arc that describes every possible 3-partition we could have through that point

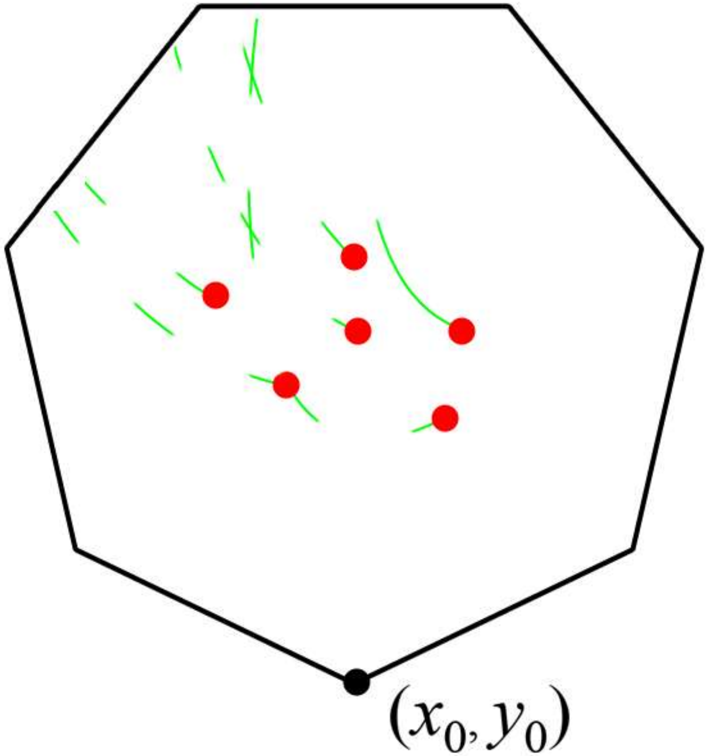
Step 3

- Construct these hyperbolic arcs for each point in the point set



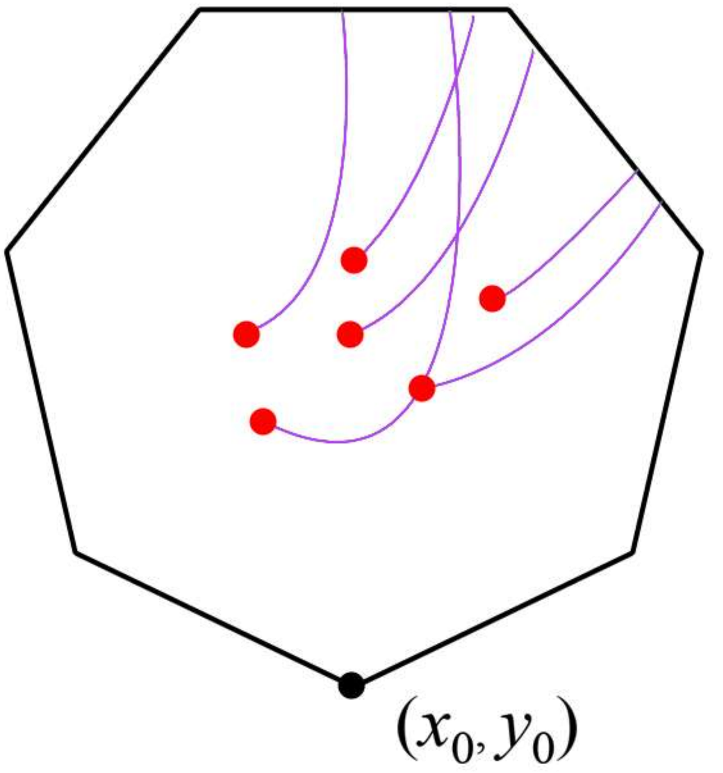
Step 3

- Construct these hyperbolic arcs for each point in the point set
- Subdivide the arcs into segments whose sector regions cut off the correct number of points (g_3)



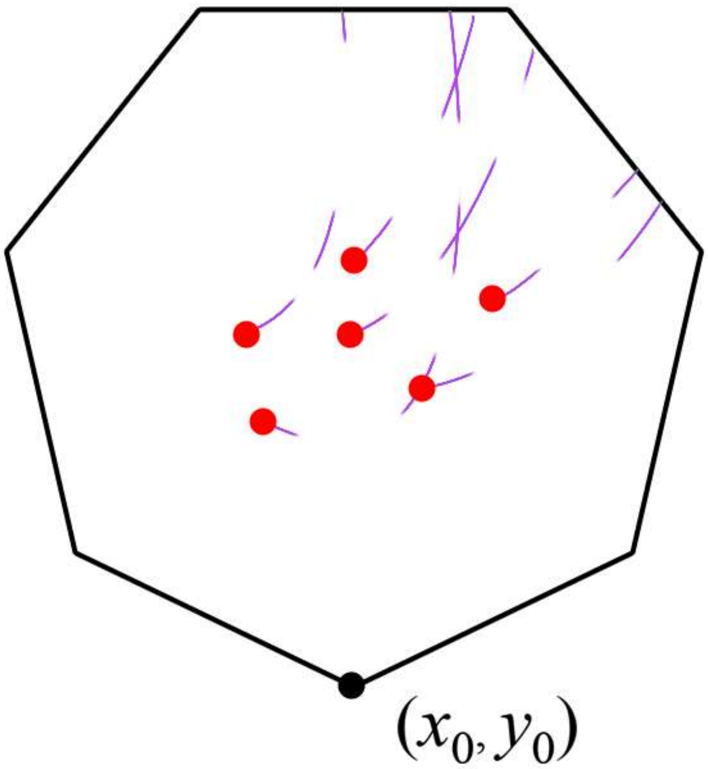
Step 4

- Repeat the same step for potential regions L, *cutting off* g_l/g of the area



Step 4

- Repeat the same step for potential regions *L*, *cutting off* g_l/g of the area



Step 5

- The hyperbolic arcs for R and L must intersect somewhere, by the geometric lemma; find this intersection and we're done