Structuralism

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1 Introduction

With the rise of multiple geometries in the nineteenth century, and in the last century the rise of abstract algebra, of the axiomatic method, the set-theoretic foundations of mathematics, and the influential work of the Bourbaki, certain views called "structuralist" have become commonplace. Mathematics is seen as the investigation, by more or less rigorous deductive means, of "abstract structures", systems of objects fulfilling certain structural relations among themselves and in relation to other systems, without regard to the particular nature of the objects themselves. Geometric spaces need not be made up of spatial or temporal points or other intrinsically geometric objects; as Hilbert famously put it, items of furniture suitably interrelated could satisfy all the relevant axiomatic conditions as far as pure mathematics is concerned. A *group*, for instance, can be any multiplicity of objects with operations fulfilling the basic requirements of the binary group operation; indeed the very abstractness of the group concept allows for its remarkably wide applicability in pure and applied mathematics. Similar remarks can be made regarding other algebraic structures, and the many spaces of analysis, differential geometry, topology, etc. Of course, mathematicians distinguish between "abstract structures" and "concrete ones", e.g. made up of familiar, basic items such as real or complex numbers or functions of such, or rationals, or integers, etc. (For example, the space L^2 of square-integrable functions from $\mathbb{R}$ (or $\mathbb{R}^n$) to $\mathbb{C}$, with inner product $(f, g) = \int f^* g \, d\mu$, where μ is Lebesgue measure, is a "concrete" example of a Hilbert space.) But it is characteristic of a thoroughgoing structuralism to treat even these systems as like the more "abstract" ones in that the "objects" involved serve only to mark "positions" in a relational system; and the "axioms" governing these objects are thought of, *not* as *asserting definite truths*, but as *defining* a type of structure of mathematical interest. In some sense to be clarified, the objects only serve as relata of key relations, and their "individual nature" is of no mathematical concern, if one can even speak of such a nature. Sometimes it is even said that such objects—"structural objects"—have only the properties and bear only the relations to other such objects as required by the relevant axioms or defining conditions laid down in the branch of mathematics in question.

Now in the course of one paragraph, we have gone from a commonplace of the modern mathematical point of view to some pretty deep-sounding issues that will require some sorting out. We have various notions of "abstract" and "concrete" to contend with. Moreover, we have the tantalizing suggestion that perhaps mathematical objects "have no nature" at all, beyond their "structural roles". Does this make sense? And we confront the question whether structuralist tenets apply at the most fundamental levels at which we speak of mathematical objects: numbers, sets, functions, even relations themselves; and if they do apply, how so? It is one thing to study algebraic, geometric, or topological structures, etc., independently of any particular objects making them up, in the sense that none are preferred, except for special purposes in particular contexts. But are we to understand talk of *the natural numbers* in the same way? And what about *collections* and *operations*, or more generally *relations* (which can include collections as unary relations)? In short, *where does structuralism start*? And then, *where does it take us*? Does it provide novel insights, perhaps even answers to, or dissolutions of, any long-standing metaphysical and epistemological problems concerning mathematics, or does it just shift these with some new terminology?

As we shall see, there are some strikingly different ways of developing the informal, intuitive ideas associated with structuralism, and a large part of our task in this paper will be to delineate and compare these alternatives along several important dimensions. As will emerge, there are a number of interesting trade-offs, and it is surprisingly difficult—perhaps impossible—to formulate a version that combines all the advantages exhibited by some version or other while avoiding the pitfalls. At this stage, it will help guide our investigation to formulate a number of central questions which any developed version of structuralism ought to answer. Here are five such:

(1) What primitive notions are assumed, and, in particular, what is the background logic? Does it go beyond first-order logic? If higher-order logic is presupposed, what is the status of relations and functions as objects? What limitation does this imply on the structuralist approach

in question?

(2) Already we have hinted that the term '*axiom*' is ambiguous, meaning either "*defining condition*" on a type of structure, in which case nothing is being *asserted,* or meaning "basic or initial assumption", as an assertion that can be true or false, rationally credible to some degree or not, and so forth. It is characteristic of a "structuralist" approach to a branch of mathematics to appeal to "axioms" in the former sense, and this is precisely what Dedekind did in his treatment of arithmetic [9], which justifies calling his approach "structuralist" at least in a minimal sense.(Similarly, this was Hilbert's conception in his famous correspondence with Frege.[13]) But then what are the *assertory axioms* of the framework in question? (To modify a good saying, "Not by definition alone!")

(3) Is there a thoroughgoing *elimination* of structures-as-objects? If not, what is a *structure*? Moreover, what is a *mathematical* structure? [21]

(4) As a special case of (2), what assumptions are made governing the *mathematical existence* of structures, and how is this understood? In particular, how, if at all, is the *indefinite extendability* of the realm of mathematical structures accounted for?

(5) How is *reference* to structures understood, and what account can be provided of our *epistemic access* to structures?

The main types of mathematical structuralism that have been proposed and developed to the point of permitting systematic and instructive comparison are four: structuralism based on model theory, carried out formally in set theory (e.g. first or second order Zermelo Fraenkel set theory), referred to as "STS" (for "set-theoretic structuralism"); the approach of philosophers such as Shapiro and Resnik of taking structures to be *sui generis* universals, patterns or structures in an *"ante rem"* sense (to be explained below), referred to as "SGS" (for "*sui generis* structuralism"); an approach based on category and topos theory, proposed as an alternative to set theory as an over-arching mathematical framework, referred to as "CTS" (for "category-theoretic structuralism"); and a kind of eliminative, quasi-nominalist structuralism employing modal logic, referred to as "MS" (for "modal-structuralism").[1] Let us take these up in turn, guided by questions (1) - (5), with the aim of understanding their relative merits and the choices they present.

[1] For further details on these types of structuralism and their comparison, see [19], [33], and [20]. For a somewhat different, but largely overlapping, way of dividing up the subject (less category theory), see [29].

2 Structuralism in Set Theory

This approach has, of course, arisen within mathematics itself and is the standard way of articulating a kind of structuralism. Mathematical structures are certain sets, order tuples of a domain together with distinguished relations, functions, and possibly individuals (members of the domain). (We take as familiar the interdefinability of functions and relations.) Structures can be *models* of certain theories, *satisfying* their theorems. Such theories would include all the ones normally encountered in mathematics. Furthermore, structures may be related by *isomorphisms*, relation-preserving bijections, or by *embeddings*, *homomorphisms*, etc., all defined in set theory in familiar ways. Various important properties of theories, e.g. consistency, completeness, categoricity, etc., correspond to or are defined by conditions on their models, a central concern of model theory.

In order to answer the questions (1) - (5), framed above, it is necessary to identify a particular set theory in which model theory is carried out. The standard choice is Zermelo-Fraenkel set theory, usually with the Axiom of Choice. But various other choices present themselves, e.g. are there to be *Urelemente* (in addition to the empty set), and—of greater significance—is the background logic to be *first-order* logic, or is it (an axiomatic part of) *second-order* logic? If proper classes are admitted, are they done so *conservatively* with respect to set-theory proper, as is the case in the (two-sorted, first-order) system NBG, or as an essential enrichment of set theory, as in the second-order system of Morse-Kelly (with impredicative class comprehension)? To some extent, the answers to questions (1) - (5) depend on these, and perhaps other, choices. But let us approach them with first-order ZFC in mind, as the central, and most common, case.

Concerning (1) and (2), then, the logic is first-order with equality, the sole non-logical primitive is ϵ, set-membership, and the non-logical axioms, in the assertory sense, are those of ZFC. (3), the nature of mathematical structures according to STS, has already been described. Structures-as-objects are just certain sets, and so they are not eliminated. There is, however, elimination of any non-set-theoretic structures, e.g. for the natural numbers, or the reals, etc., as *sui generis* objects, e.g. as Dedekind viewed them.[10] On STS, one identifies such objects in a particular way, but this is for convenience, e.g. for a smooth development of ordinal number theory, etc. But it is recognized that the branch of mathematics in question (number theory, real analysis, etc.) concerns any structures of the relevant isomorphism type.[2] On

[2] Equivalence in somewhat weaker senses than isomorphism may of course be

this view, a question such as, "What are the natural numbers, really?", is dismissed as misguided. As to the question, what distinguishes "mathematical" from other structures, a natural answer is forthcoming, viz. that, typically, there is vocabulary (predicates, functional expressions) underlying given structures and that the source of that vocabulary indicates the field of inquiry, e.g. space-time physics, as opposed to purely mathematical geometry, although in a field such as mathematical mechanics, the line may not be sharp, especially if the theory in question is being explored not as an actual description but as merely a theoretical possibility.

Turning to question (4), one virtue of STS is the clarity of its standard of mathematical existence of structures, as this just means existence as sets in ZFC. Of course, many questions are not answered by these axioms, but for the most part these concern *extraordinary* mathematics, e.g. large cardinals, not the structures and spaces of ordinary mathematics, for which a system such as ZFC is more than adequate. What about the issue of extendability? While there is a built-in indefinite extendability of structures in the strict, set-theoretic sense (as the ordinals "go on and on"), there is a certain crucial limitation when it comes to structures for set theory itself. Here we encounter a massive exception to the structuralist point of view, in that, on its face-value interpretation, set-theory itself is *not* treated structurally: its axioms are not understood as defining conditions on structures of interest but are taken as assertions of truths in an absolute sense. One speaks of "*the* cumulative hierarchy", or even "the real world of sets". *In* first-order ZFC, of course, no such thing is officially recognized, although in NBG or in second-order ZFC, one has "the universe" as a proper class. Clearly these latter theories explicitly violate *extendability* as a general principle, but first-order set theory as practiced violates it as well, as it implicitly recognizes "the sets" if only in a plural sense, as the very subject matter of the theory. Why should such higher-type totalities not be a subject of mathematical investigation? Why should not higher-order "collections" be recognized, i.e. what *prevents* the "collectibility" of "all sets", in a logical sense inherent in set theory itself? And why aren't such collections subject to operations analogous to those of set theory itself, including formation of singletons, power collections, etc.? (Of course, this latter question

recognized, allowing for definitional extensions of given structures, e.g. adding addition and multiplication to successor in second-order arithmetic. Note that model theory can treat second- and higher-order theories and their models as objects, even if the background logic for the set theory is first-order. Also, note that our talk of "isomorphism types" is only a manner of speaking. In first-order ZFC, the relevant types cannot exist, as they would be proper classes.

applies to theories of proper classes as well, which is one of the reasons they are an embarrassment to set theory.)

As to question (5), reference to structures is just a special case of reference to sets, and this is usually just taken for granted. Once given a starting point, the null set, we know what its singleton is, what the pair of it and its singleton is, what the finite von Neumann or Zermelo ordinals are, what the hereditarily finite sets built on the null set are, what the power set of this is, etc., etc. Whether all these things, as intended—and this includes full power sets (containing *all* subsets of given sets) and enough ordinal levels to satisfy the Replacement axiom— actually occur in "the real world of sets" and how we can know are esoteric questions that surely no mathematician *qua* mathematician would bother about.

As this brief overview indicates, STS, despite its clarity and definiteness, is not without its problems. Three in particular stand out:

(i) As already indicated, set theory itself is not treated structurally, and this despite the fact that there are multiple set theories around, all legitimate as avenues of mathematical investigation. Thus, STS is deprived of one of the most natural ways of "justifying axioms" from a mathematical point of view, that is by appeal to our interests in a particular kind of structure. For example, consider well-foundedness (Axiom of Regularity): rather than "defend" this as *true* in some absolute sense, a structuralist would simply cite our interest in exploring well-founded sets, without denying that non-well-founded sets "exist" or are a fit mathematical subject. Similar remarks can be made about the axiom of Replacement. It need not be guaranteed by "the meaning of 'set"' or by some *a priori* insight into "set-theoretic reality"; it is sufficient as a coherent condition on *domains* or *universes* of sets, guaranteeing their *largeness* relative to any of the members. But such an answer is not available to STS (for why should the real world of sets be large?). In short, this view fails to block bad questions about set theory.

(ii) STS is saddled with a maximal totality (or plurality) of sets, hence of structures, in conflict with intuition and with mathematical practice, which always has the potential of transcending any domain proposed as a limit. It violates a truly general extendability principle.

(iii) It confronts mathematically esoteric and seemingly intractable ontological and epistemological questions, e.g. in connection with the null set, with "full power sets", with "the order type of the universe", and so forth.

Thus, despite the great successes of model theory, considerations such as these have motivated some mathematicians, logicians, and philosophers to seek alternative ways of articulating structuralist ideas.

3 Structures as *Sui Generis* Universals

Like STS, this approach conceives of structures as *absolute objects*, but not as abstract particulars but rather as universals, *patterns* in Resnik's terminology, answering to "what all particular systems of a given type—whether made up of concreta or abstracta such as sets—have in common". Patterns are thus reified; they *are* the types. One thus speaks literally of "*the* natural number structure": its constituents, the numbers, are not sets but are conceived as mere "places" or "positions" defined by the structural relations, themselves determined by mathematical axioms or conditions. To be the number *two*, for instance, is just to be the third place in this natural number structure (if we begin, with Frege, with 0). Since places are here treated as objects in their own right, rather than as "offices" to be filled by particulars, Shapiro uses the term "*ante rem*" for these structure types, in contrast to the usual *in re* structures of set theory or concrete realizations. Many of the latter are instances of a unique one of the former.[3] Thus, SGS is not eliminativist at all. So long as some mathematical conditions—normally given as statements in second-order logic—are *coherent*, there will be an *ante rem* structure which they automatically describe. The various number systems, for instance, can be instantiated multiply by sets, but—in answer to Benacerraf's puzzle—no such instantiation is the true *ante rem* number structure. It is to such a structure that our ordinary designators actually refer, and set-theoretic reductions are merely convenient representations. Since *ante rem* structures are conceived as abstract types standing "above" already abstract instances (e.g. of sets) and moreover as instantiating themselves, i.e. as made up of objects fulfilling the structural defining conditions, the term "hyperplatonist" seems apt for this view.

In addressing questions (1) - (5), we will follow Shapiro's presentation which outlines a formal framework.[4] Concerning (1) and (2), Shapiro assumes a second-order background language, assumed "to include a rudimentary theory of collections" and the machinery for speaking of functions and relations among *places* in *structures*. It is natural to understand by this an axiomatic second-order logic including the usual unrestricted (impredicative) comprehension scheme for collections and relations. A system of axioms—structure theory—governing the primitives, *structure* and *places*, is presented: these include axioms very much like those of second-order ZF, including axioms of *Infinity*, *Powerstructure*, and *Replacement*. In addition, there is a *Coherence* axiom: "If Φ is

[3] Parsons [28], citing Tait, has described the thought process of moving from particular *in re* realizations to the *ante rem* structure as "Dedekind abstraction".

[4] Alternative presentations, such as Resnik's, which is more informal, may not be entirely accurately represented by Shapiro's formulation.

a coherent formula in a second-order language, then there is a structure that satisfies Φ." Here 'coherent' is a further primitive, corresponding to model theory's notion of *satisfiability. (*As Shapiro points out, it is not reducible to a formal condition such as consistency or even ω-consistency.) *As a direct consequence of this axiom, the basic distinction introduced above between axioms-as-defining-conditions and axioms-as-assertions collapses: any conditions that might possibly be satisfied are in fact satisfied,* i.e. *are true of* an *ante rem* structure, but since the axioms are understood to be *about* such a structure, according to SGS, they are *true simpliciter.*

In addition to these axioms, a (second-order) Reflection principle can be assumed, guaranteeing large structures (corresponding to strongly inaccessible cardinals).

Turning to (3), its main part on the conception of structures as objects has already been addressed. And presumably we have also addressed the subsidiary question of distinguishing *mathematical* structures from others: it is natural to take these to be structures specified by coherent second-order conditions in the language of structure theory. Of course, only a very special subcollection of these would be of genuine mathematical interest, and it would surely be misguided to attempt any general characterization of these; but physical and other non-mathematical structures would presumably not be found, insofar as their description requires vocabulary beyond that of pure structure theory (second-order logic plus the primitives 'structure' and 'place').

As to (4), existence of structures is of course spelled out in the axioms of structure theory. What about extendability? Clearly SGS improves on STS in applying to set theories, without having to recognize any one as "the one true one", and so it avoids any maximal plurality or collection of sets. Still, there is a maximal collection of *places in structures* as a consequence of the second-order comprehension principle which structure theory presupposes. And while there is no collection of all *structures*, as this would be a third-order object, still *there are*, speaking plurally, *all the structures*, i.e. an inextendable "universe of structures", informally speaking, whether or not the formalism allows talk of such a collection.

Finally, regarding (5), as indicated, reference to *ante rem* structures and their places is just supposed to occur when we gain mastery over the relevant mathematical vocabulary. Just as in learning the English alphabet, we learn to refer to the letters as types, so we learn to refer to the natural numbers, and then, somewhat later, to the rationals, the reals, and maybe even the complexes.

How does SGS fare with respect to the problems (i) - (iii) affecting

STS, described above? (i) is overcome in SGS, as it treats set theory (theories) structurally like other mathematical theories. Sets are not conceived of as abstract particulars, existing in their own right and forming the subject matter of mathematics. Rather, like numbers or elements of algebraic structures, they are conceived merely as places in an abstract structure. The multiplicity of set theories fits nicely into the view; for instance, well-founded sets are realized in one *ante rem* structure (actually in many such), but there are others realizing non-well-founded sets. The "membership relation" of the latter simply behaves differently. Likewise with regard to other structure-characterizing axioms such as Replacement. In sum, SGS succeeds in blocking a number of bad questions about set theory where STS only encouraged them. Similarly, the puzzles for STS under (iii) do not arise for SGS. There is no mystery about the null set, for example. This is just a structural "starting point" in an *ante rem* structure for set theory, an object to which no other place bears the relevant two-place relation (ordinarily called "membership"). Moreover, there is no question about "real-world power sets", as such are not distinguished. So long as "full power set" (of a given infinite set) is coherent, an *ante rem* structure will have a place for such (as places). Of course, there are also many structures with less-than-full power sets, but that does not matter. The question of coherence can certainly be raised (and typically is by predicativists), but the issue is not an ontological one, according to SGS. This seems right. Similarly, there is no problem about "the order type of the universe", as no "universe" of all sets need be recognized. Many universes are possible, hence actual as *ante rem* structures. If power sets are full and second-order Replacement holds, the order type of such a universe will be a strongly inaccessible cardinal, but, presumably, there are boundlessly many of these.

At first blush, it seems that SGS also handles problem (ii) quite deftly, as there simply is no maximal totality of sets forming an *ante rem* structure. As far as these structures for set theory go, the extendability principle is respected. However, as already pointed out in connection with point (4) above, SGS is still saddled with maximal totalities of its own, namely the class of all places (explicitly delivered by second-order logical comprehension in structure theory) and the plurality of all *ante rem* structures, implicit in the overall system with the Coherence axiom. In effect, SGS has merely traded "the totality of sets" for "the totality of structures", and so does not endorse the general extendability principle.

There are, moreover, further problems that have been raised against SGS, problems that do not confront STS. Two interrelated such problems should be mentioned here. (Our numbering continues that of the list of problems begun above.)

(iv) *Places* as objects in *ante rem* structures raise questions concerning their very identity.[5] It seems, from a structuralist perspective, that intra-structural relations alone should suffice to distinguish these objects, without appeal to external relations or individual constants. But then places in a structure should satisfy a Leibnizian principle of *identity of structural indiscernibles:* any items bearing exactly the same intra-structural relations to other items should be not many but one.[6] But this immediately implies that the structure in question must be *rigid,* i.e. admitting no non-trivial automorphisms. While that is true of some key mathematical structures, e.g. the natural numbers, the field of real numbers, segments of the cumulative hierarchy of sets, etc., non-rigid structures nevertheless abound in mathematics, e.g. the complex numbers (interchanging i and $-i$), the additive group of integers (interchanging +1 and -1), geometric figures with reflectional symmetry, homogeneous Euclidean n-space, etc. While it is true that all these non-rigid structures can be recovered inside rigid ones (via reduction to sets), that seems counter to the whole thrust of SGS. Another reply suggests simply not seeking any criterion of identity for *places.* The debate over this continues.[7]

(v) Are *purely structural relations* intelligible in the context of putatively structural objects—*places*—as relata? This objection challenges the very notion of *ante rem* structure, whether rigid or not. If we do not appeal to the relata of a structure as somehow independently given, e.g. invoking reference by singular terms in our background language as standard platonism does, but as determined by structural relations—which is surely part and parcel of the SG view—what have we to go on in specifying structural relations other than the axioms (defining conditions) themselves?[8] But, as Russell pointed out in his criticism of Dedekind's early expression of SGS [9], [10], the axioms don't distinguish any particular realization from among the many systems that satisfy them.[9] What, for example, can it mean to speak of "*the order-*

[5] This objection has been raised independently by Keränen [22] and Burgess [8].

[6] Note that this is weaker than the claim, sometimes made, that places in *ante rem* structures have only the properties—or only the essential properties—required by the axioms defining the structure in question. These latter conditions seem impossible to realize: for example, surely the natural numbers (according to SGS) are abstract, non-spatio-temporal, non-physical, etc., even essentially so; yet the axioms are silent on such matters.

[7] See Shapiro's "Structure and Identity", Keränen's "The Identity Problem for Realist Structuralism II", and Shapiro's "The Governance of Identity" in [23].

[8] As Parsons put it in assessing the idea of structural objects as "incomplete objects", this view is "itself incomplete for it neglects the fact that the *relations* of a structure are themselves given only by formal conditions..." [28], 334).

[9] As Russell wrote, "It is impossible that the ordinals should be, as Dedekind

ing" of "*the natural numbers*" as objects of an *ante rem* structure unless we already understand what these numbers are *apart from their mere position in that ordering*? Surely the notion of *"next"* makes no sense except *relative to an ordering or function or arrangement of some sort,* something Dedekind was careful to take into account when describing *simply infinite systems*, which always involve objects "set in order by transformation φ". (Clearly anything whatever can be "next after" anything else in some system or other.) Thus the notion of an *ante rem* structure seems to involve a vicious circularity: such a structure is supposed to consist of *purely structural relations among purely structural objects, but understanding either of these requires already understanding the other.* Whereas the Keränen-Burgess objection granted the relations and raised questions about how these alone could determine the objects (unless the structures are rigid), this objection questions such talk of relations in the first place, and thereby the very notion of "Dedekind abstraction" which is supposed to lead to them.[10]

In fact, SGS, seems ultimately subject to the very objection of Benacerraf [4] that helped inspire recent structuralist approaches to number systems in the first place. Suppose we *had* the *ante rem* structure for the natural numbers, call it $< \mathbb{N}, \varphi, 1 >$, where φ is the privileged successor function, and 1 the initial place. Obviously, there are indefinitely many other progressions, explicitly definable in terms of this one, which qualify equally well as referents for our numerals and are just as "free from irrelevant features"; simply permute any (for simplicity, say finite) number of places, obtaining a system $< \mathbb{N}, \varphi', 1' >$, made up of the same items but "set in order" by an adjusted transformation, φ'. Why should

suggests, nothing but the terms of such relations as constitute progressions. If they are to be anything at all, they must be intrinsically *something*; they must differ from other entities as points from instants, or colors from sounds...Dedekind does not show us what it is that all progressions have in common, nor give any reason for supposing it to be the ordinal numbers, except that all progressions obey the same laws as ordinals do, which would prove equally that *any* assigned progression is what all progressions have in common..." [31], p. 249.

[10] In correspondence, Shapiro has pointed to "finite cardinal structures"—strucures with finitely many distinct "places" but no relations at all (other than (non-)identity)—as showing that places should not be thought of as dependent on structural relations. Neither places nor relations are prior to the other. But such structures seem an ultimate offense against Leibnizian scruples. For what distinguishes one "place" from another? How can we even make sense of mapping the places to or from the many finite collections such a structure is supposed to exemplify (e.g. all pairs, or triples, or quadruples, etc.)? Does it even make sense to think of labelling these "things"? In the case of identical bosons of quantum mechanics (a famous case where labelling is problematic), we ultimately have the option of dropping the notion of two particles, say, in favor of a "boson-pair system". But such a move in the case of finite cardinal structures would destroy their cardinality (at least for any $n > 1$)!

this not have been called "the archetypical *ante rem* progression", or "the result of Dedekind abstraction"? We cannot say, e.g., "because 1 is really *first",* since the very notion "first" is relative to an ordering; relative to φ', $1'$, not 1, is "first". Indeed, Benacerraf, in his original paper, generalized his argument that numbers cannot really *be* sets to the conclusion that they cannot really *be* objects at all, and here, with purported *ante rem* structures, we can see again why not, as multiple, equally valid identifications compete with one another as "uniquely correct". Hyperplatonist abstraction, far from transcending the problem, leads straight back to it.

Other problems can be raised for SGS, in particular problems concerning the Coherence axiom and the primitive "coherent". It can be thought of as a post-Gödelian substitute for formal consistency: the axiom mimics Hilbert's idea that consistency suffices for mathematical existence. But of course it is not a formal notion and seems no clearer than a primitive notion of (second-order) logical possibility, indeed perhaps less so, for do we have anything as developed as modal logic governing "coherent"? And if we identify these notions, then the Coherence axiom appears even more problematic, for why should mere logical possibility suffice for existence? Indeed, why not just rest with the former, which, as we shall see below, enables avoidance of the very maximality problems that plague both STS and SGS?

For all these reasons, then, we are motivated to look at some alternative, non-absolutist approaches to structuralism.

4 Structuralism in Category Theory

Like set theory, category theory ("CT") has arisen within mathematics both as a branch of mathematics making its own special contributions (in the case of category theory, to algebraic topology and algebraic geometry), and also as a general framework or setting for vast amounts of mathematics. Mac Lane and others have taken the further step of proposing topos theory as providing an alternative foundation for mathematics, comparable to and in some ways superior to set theory, and, more recently, Awodey has explicitly suggested that category theory provides a natural framework for realizing structuralism, one that philosophy of mathematics should employ and pursue.[11]

The basic idea behind Awodey's suggestion is that category theory provides an effective way of codifying and illuminating mathematical structure through its focus on families of structure-preserving mappings

[11] In this discussion, we will presuppose a familiarity with the basic notions of 'category' and 'topos'. Awodey [1] presents an accessible overview, as do Mac Lane[24] and Bell [2]. See also McLarty [27].

between objects having the relevant kind of structure and through functorial relations between different categories, which exhibit important relationships among different kinds of structures, of special interest in advanced mathematics. Moreover, the CT approach to mathematical structure has advantages over the Bourbaki (set-theoretic) approach, in that descriptions via mappings (morphisms, also functors) abstract from the initial means by which structures of a given type may have been introduced. Topological structure, for example, is determined via continuous maps to and from other spaces regardless of the details of their original introduction, via open sets, limit points, closure operations, etc. Indeed, it is remarkable how much of what are normally thought of as operations and relations *internal* to a structure can be recovered via morphisms to and from other "objects" (what would normally be called "structures", formally treated as point-like in CT). It is even possible, for example, to get the effect of elementhood itself, via morphisms between a terminal object and a given object (where an object 1 is *terminal* just in case, for any object C in the category there is a unique morphism from C to 1). Moreover, CT constructions exhibit in a precise way how mathematical structure is significant only "up to isomorphism", and they bring out "shared structure" through novel methods of generalization (e.g. via universal mapping properties, e.g. of products, generalizing the notion of Cartesian product in set theory).

Clearly, CT provides important insights into various kinds of mathematical structures and into "mathematical structure" in a general sense. But does it provide a genuine structuralist framework for mathematics on a par with STS or SGS? In assessing this, some basic distinctions must be born in mind. First, it is essential to distinguish between CT *as mathematics* in its own right and CT *as a foundational framework*. Texts on CT as mathematics typically make reference to a background set theory or universe of sets relative to which CT is to be understood and carried out.[12] If generally enforced, this would make CT dependent on set theory, and it could not be considered to provide an autonomous framework. Thus, the issue of autonomy *vis-à-vis* set theory must be addressed. In particular, Mac Lane has proposed that mathematics generally can be developed in a certain kind of topos (satisfying a condition called "well-pointedness", leading to extensional discrimination of morphisms as in classical mathematics), and that this is an alternative to set-theoretic foundations. But then it must be possible to make sense of such topoi without falling back on set theory, either conceptually or ontologically. Similarly, Bell has proposed that any topos with a natural number object can serve as a background universe for (ordinary)

[12]See, for example, Mac Lane and Moerdijk [25], Bell [3], Freyd and Scedrov [14].

mathematics, and that this results in an interesting relativity of ordinary mathematical concepts (e.g. "real number", "continuous function on reals", etc.) to background topos (whose internal logic is in general intuitionistic, but which may be classical if stronger conditions, such as well-pointedness, are built in). Some core mathematics turns out to be invariant over the relevant totality of topoi, but many things are not. To sustain this local, relativist view, again it must be possible to carry out the mathematics and metamathematics of topoi without falling back on set theory.

A second important distinction concerns the term "category theory" itself. On the one hand, there are the so-called "first-order axioms on categories", the standard conditions interrelating the domains and codomains of morphisms with the binary "composition" operator and governing identity morphisms. To these may be added a variety of further conditions defining various kinds of topoi: for "elementary topoi", the basic requirements generalizing Cartesian products, functional exponentiation, and set theory's apparatus for (Boolean) propositional functions ("subobject classification"); for foundational purposes, further conditions guaranteeing a natural number object, and stronger principles such as an axiom of choice, well-pointedness, etc. It is clear, however, that these are not "theories" in the sense in which ZFC is a theory, for these "axioms" on categories and topoi are merely defining conditions, simply telling us what first-order conditions must be satisfied by anything to qualify as a category or a topos. Thus, the primitives ('domain', 'codomain', 'composition') need not have any particular meanings, and these axioms *assert nothing* by themselves. As Awodey himself puts it,

> "a category is *anything* satisfying these axioms. The objects need not have 'elements', nor need the morphisms be 'functions', although this is the case in some motivating examples...We do not really care what non-categorical properties the objects and morphisms of a given category may have..." ([1], 213.)

Thus, categories and topoi are conceived in the manner of algebraic structures such as groups, rings, etc., and the informal "theory of categories", which of course contains genuine assertions including foundational ones *about categories and topoi* must not be identified with these first-order "axiom systems", but must somehow be rich enough to express substantive claims about structures satisfying these axioms. This is essential in assessing the status of CT *vis-à-vis* set theory. For example, just because notions of 'collection' and 'operation' are not used in the first-order defining conditions does not mean that they are really bypassed in theorizing about categories in the relevant sense.

This is an appropriate place to recall an early critique by Feferman of Mac Lane's claims on behalf of CT as providing a foundational alternative to set theory [11]. In essence, Feferman argued that category theory presupposes and uses, informally, notions of *collection* and *operation*, both in saying what a category (or topos) is, and in relating categories to one another through *homomorphisms* or *functors*. Moreover, a foundational framework for mathematics must provide some systematic account of these notions, something set theory does but category theory does not. It was explicitly recognized that alternatives to standard set theory might also provide this, so that the claim was not that CT depends on set theory *per se,* but rather that, as it stands, it is inadequate.

Now there is a temptation to respond to this by taking CT to be a theory of families of functions related by composition, and to claim that this in principle is no different from set theory as a theory of sets (and possibly individuals) related by membership. The notion of "function" is one that mathematicians use all the time, and even set theory can be derived from an analogous theory of functions, as was originally carried out by von Neumann [34]. CT could be viewed as an alternative, perhaps more interesting, systematic approach based on the familiar notion of function. What's wrong with that?

The problem is that this depends on a special, privileged interpretation of 'composition', hence of 'morphism', in diametric opposition to the algebraico-structuralist reading that category theorists apply to their own systems (cf. the quotation from Awodey, above). Moreover, the "axioms" defining categories and topoi are silent on the matter of mathematical existence of these structures, leaving unresolved basic questions concerning their sizes and their scope. (E.g., is there really a category of *all* categories? If so, how is Russell's paradox avoided?, and so forth.)

Thus we see that an algebraico-structuralist reading of the CT axioms actually underlay Feferman's critique;[13] moreover, it simply vitiates the above response, which, in any case, is inadequate as just indicated.

Turning back to questions (1) - (5) we have been putting to each version of structuralism, we find ourselves now in the awkward position of

[13] As Feferman originally wrote,

"...when *explaining* the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc., we implicitly *presume as understood* the ideas of *operation* and *collection*; e.g. we say that group consists of a collection of objects together with a binary operation satisfying such and such conditions." [11], 150.

being unable to answer any of them with any definiteness, save perhaps (3) concerning elimination of structures as objects. (For future reference, let us number this problem of underspecification (vi).) Consider (1) on primitives and background logic. Of course we know what the primitives of the first-order CT axioms are, but the question is not about the definition of "category", but rather about the primitives of the background (informal) substantive mathematical-foundational (meta-)theory, which, as Feferman observed, employs notions of *collection* and *operation* and *functor.* The fact that these are not among the primitives of the definitional CT or topos-theoretic axioms is irrelevant. We cannot even say that the background logic is to be first-order, nor that it is intuitionistic, like the *internal* logic of a topos. Almost certainly classical logic is required for some of the results of category theory as substantive mathematics. Bell's approach, for example, which treats topoi as models of "local set theories", requires some substantial metalogic. The languages of local set theories are type-theoretic, but with no cardinality restrictions on the type symbols that may occur. (Bell even acknowledges at the outset that for some purposes, a background set theory such as NBG may be needed.)

The situation regarding (2) is no better. We are simply not in a position to identify agreed upon substantive (assertory) axioms explicitly or implicitly assumed in category and topos theory *qua* genuine mathematics or metamathematics, without falling back on set theory, that is.

Presumably, structures-as-objects are not eliminated in CT structuralism, for they make up one of the sorts (the *objects*) of typical categories, e.g. of groups, of vector spaces, of topological spaces, etc.[14] This is a short answer to the first part of (3), but what about the second and third, concerning the nature of structures and what distinguishes *mathematical* from other structures? As objects inside categories, structures are treated as "pointlike", and everything is expressed in terms of morphisms and functorial relations, leaving the nature of structures themselves up in the air. While it is true that CT has the resources to express its own account of various spaces, e.g. topological, one wonders whether such an account would be intelligible without prior acquaintance with ordinary set-theoretic constructions, e.g. via open sets.

The first part of (4), of course, as a special case of (2), remains unanswered. Questions of mathematical existence of structures, including

[14]Even this point, however, is somewhat debatable, as the objects of a category can actually be dispensed with in favor of special morphisms, identity morphisms. In any case, morphisms carry structure, and clearly *they* are not eliminated, so that in a broad sense, distinctively mathematical objects are recognized.

categories and topoi, are usually deferred to an unspecified background set theory or model thereof, but, as already said, this is appropriate only when pursuing CT as pure mathematics, but surely not when proposing it as a foundation or structuralist framework. Concerning the second part of (4), extendability, this principle is certainly in the spirit of CTS, as Mac Lane's remarks on the open-endedness of mathematics attest.[15] But we saw above that SGS, while critical of "the real world of sets", nevertheless runs into violations due to second-order logic, and it remains to be seen how CTS can avoid such problems. As to (5), how CTS contributes to our understanding reference and epistemic access to structures, we remain quite in the dark.

Even without enough development to answer these questions, CTS does appear to escape the problems (i) - (iii) affecting STS and (iv) and (v) affecting SGS, on the proviso that it can avoid commitment to maximal totalities. Clearly it treats set theories structurally along with other branches of mathematics. The null set, for example, is treated simply as an initial object of a category, not as a mysterious empty abstract container. And there need be no commitment to any unique set-theoretic reality. Nor is CTS plagued by the special objects and relations of *ante rem* structures.

On the positive side, category theory has much to offer by way of insights into mathematical structure and through its novel approach to generalization via morphisms and functors. Equally clearly, however, in light of the foregoing discussion, it is inadequate as a foundational framework as it stands. If it is not to fall prey to a dependence on set theory after all, some alternative way of developing it which is responsive to the above basic questions must be found. In the final section below, we will mention one such alternative.

5 Modal-Structuralism

A good way into this approach (developed in [17] and [18]) is via Russell, who wrote early on in [32],

> It might be suggested that, instead of setting up '0', 'num-

[15] Thus, he writes,

"Understanding Mathematical operations leads repeatedly to the formation of totalities: the collection of all prime numbers, the set of all points on an ellipse...the set of all subsets of a set..., or the category of all topological spaces. There are no upper limits; it is useful to consider the "universe" of all sets (as a class) or the category *Cat* of all small categories as well as CAT, the category of all big categories. After each careful delimitation, bigger totalities appear. No set theory and no category theory can encompass them all—and they are needed to grasp what Mathematics does." ([24], 390.)

> ber', and 'successor' as terms of which we know the meaning..., we might let them stand for *any* three terms that verify Peano's five axioms. They will then no longer be terms which have a meaning that is definite though undefined: they will be 'variables', terms concerning which we make certain hypotheses, namely, those stated in the five axioms, but which are otherwise undetermined. ...our theorems...will concern all sets of terms having certain properties. (p.10.)

No sooner had Russell offered this suggestion than he retracted it for what we today would regard as quite spurious reasons (the *definition* did not provide for *existence* of models (this from a critic of the ontological argument!), and you could not account for ordinary counting on the proposed interpretation),[16] and pursued the Fregean absolutist line (cardinal numbers as definite classes of equinumerous concepts or classes), only to encounter various problems and paradoxes leading to the abandonment of classes. In the end, we find this remarkable suggestion concerning logical and mathematical propositions generally:

> We may thus lay down, as a necessary (though not sufficient) characteristic of logical or mathematical propositions, that they are to be such as can be obtained from a proposition containing no variables...by turning every constituent into a variable and asserting that the result is always true or sometimes true. ...logic (or mathematics) is concerned only with *forms...*" ([32], p. 199.)

When we consider that *relations (propositional functions)* as well as individuals count as "constituents" of propositions, then we realize that this criterion is met by formulating mathematics in higher-order logic without constants. In fact, second-order logic suffices. In the case of number theory, for example, an ordinary statement A naturally goes over to a conditional of the form,

$$\forall R[\mathrm{PA}^2 \rightarrow \mathrm{A}](\mathrm{S}/R),$$

[16] On the eliminativist strategy suggested, counting would be understood, roughly speaking, as indicating, with numerals or related symbols, a one-one correspondence between the enumerated items and an initial segment of any progression (or any that there *might be*, on a modalized version). One can even put it constructively: counting provides a means of building a one-one correspondence between the enumerated items and a segment of any given progression. On such an account, *standing in such correspondences* plays the role that *membership* plays on the Frege-Russell account (i.e. the enumerated class (or concept) *belongs to* a Frege-Russell number).

in which 'PA^2' stands for the conjunction of the (Dedekind-) Peano axioms and 'S/R'indicates systematic replacement of the successor constant with the relation variable 'R' throughout. (Here '0' can be dropped as it is definable from successor.) If A is logically implied by the axioms, this is a truth of second-order logic; if not the result of replacing 'A' with '$\neg$A' is, in light of Dedekind's categoricity theorem.[17] Even better, taking the predicate 'number' into account as suggested, and generalizing, we obtain,

$$\forall X \forall R[\mathrm{PA}^2 \rightarrow \mathrm{A}]^X(\mathrm{S}/R).$$

where the superscript indicates relativization of all quantifiers to domain X. Thus, Russell has come full circle (at least "up to negation"!), for this is just the general interpretation "through variables" that he had suggested and dismissed at the outset. (But no matter. Russell eventually got there.) It is already a kind of structuralist interpretation, expressing that truths of arithmetic are what hold in any progression whatever. Formulated as it stands, however, it is inadequate, for suppose there *are no* progressions; then all such conditionals are vacuously true, regardless of the content of A. A better plan is to construe the generalization "always true" modally, i.e. as meaning "in any progression there might be, logically speaking", prefixing the last displayed formula with a necessity operator ('$\Box$'). Then to avoid vacuity, we may categorically lay down,

$$\Diamond \exists X \exists R[\mathrm{PA}^2]^X(\mathrm{S}/R),$$

affirming the logical possibility of a progression, which of course is compatible with the actual absence of any. The same plan works for the real number system, the complexes, for cumulative hierarchies of sets (characterized by cardinality of *Urelemente* and ordinal height), and, indeed, for any mathematical structure categorically characterized by second-order axioms. (For non-categorical theories, as e.g. in abstract algebra, the result is incompleteness, but that is as it should be. Of course, more specific types of structures, e.g. transitive permutation groups of a given order, etc., can be treated.) So far we have the basic plan of modal-structural interpretations of mathematical theories (the conditionals forming the "*hypothetical component*", the possible existence claims forming the "*categorical component*").

[17] The question may be raised here, in what background theory this categoricity theorem is proved. It need not be anything so strong as impredicative set theory. Indeed, as shown in [12], essentially only a weak axiomatic fragment of weak second-order logic is really required, in which quantification over finite subsets of the domain is taken as given. More precisely, the theorem is recovered in "EFSC", an elementary theory of finite sets and classes of individuals, conservative over Peano Arithmetic.

The background logic is second-order, with quantified S-5 modal logic, without the Barcan formula.[18] However, care must be taken in formulating second-order logical comprehension. In a modal context, unrestricted comprehension leads to intensions, transworld classes and relations. For example, suppose the predicate 'planet' is available; then we would generate a class not merely of existing planets, but of those together with "any that *might have existed*". That is we would be recognizing *possibilia*. (Notice that this follows even if the class quantifier is understood as a plural quantifier.) In general, we would be quantifying over relations, not merely relating objects in the actual world or in any hypothetical one being entertained, but *across* worlds.[19] In particular, we would generate a universal class of all *possible* objects, and corresponding universal relations among possibilia, directly violating the extendability principle (modally understood, appropriately, as "Any totality there might be might be extended"). Ordinary mathematical *abstracta* seem tame compared to such extravagances; indulging them would deprive MS of much of its interest as a distinctive program. To avoid such commitments, therefore, an extensional version of comprehension is chosen:

$$\text{(Logical Comp)} \qquad \Box \exists R \forall x_1 ... \forall x_n (R(x_1 ... x_n) \leftrightarrow \Phi),$$

where Φ lacks free 'R' and is also modal free. Note that the universal quantifiers are not boxed. In effect, (n-tuples of) individuals form collections and relations only *within* a world, not across worlds. (Officially, of course, neither worlds nor *possibilia* are recognized.)

This leaves us, however, with a level of abstract classes and relations or Fregean concepts as second-order entities. This can be dispensed with, however, by appealing to a combination of mereology and plural quantification, provided the possibility of an infinity of individuals is assumed. This itself can be expressed using mereology and plural quantifiers, for example, by

(Ax ∞) "There are some individuals one of which is an atom

and each of which combined with an atom not part of it

[18] That is, the inference from $\Diamond \exists x \varphi$ to $\exists x \Diamond \varphi$ is to be avoided.

[19] Note that it is *quantification* over transworld relations that is to be avoided. There is nothing to prevent us from applying *particular predicates* to entertain relations among given objects and "others that there might have been", as when we say innocous things such as, "There might have been a horse larger than any existing one", etc. This does commit us to possible horses as objects, nor, of course, to worlds containing such beasts.

is also one of them",

where an atom is an individual without proper parts. Given this, one can get the effect of ordered pairing of arbitrary individuals,[20] and this effects a reduction of polyadic second-order quantification to monadic, which itself can be interpreted plurally. *Thus, the relation quantifier in the above second-order comprehension scheme can be replaced with a class quantifier.* Also Φ can contain the part-whole relation of mereology.

Completing the core system is a "comprehension" scheme of mereology itself, guaranteeing the whole ("sum", or "fusion") of any individuals satisfying a given (non-null) predicate:

$$(\textstyle\sum\text{Comp}) \qquad \exists x\Psi(x) \rightarrow \exists y\forall z[y \circ z \leftrightarrow \exists u(z \circ u \,\&\, \Psi(u))],$$

where '$\circ$',"*overlaps*", is defined via "*part of", $<$*, as $x \circ y \leftrightarrow \exists v(v < x \,\&\, v < y)$, and where in Ψ, monadic second-order variables are allowed, free or bound. Thus formulated, MS is ontologically neutral to this extent: any objects whatever may stand in relevant structural relationships so long as it makes sense to speak of wholes, or of pluralities, of them, which is all that this machinery requires. In particular, a physical or spatio-temporal interpretation of "part-whole" is not required.[21]

This framework turns out to be surprisingly powerful. On the assumption of (the possibility of) just a countable infinity of atoms, not only can the modal existence of progressions (or $\mathbb{N}$-structures) be derived, but by repeated use of plurals and mereology, full, polyadic classical third-order number theory, equivalently second-order analysis, can be recovered. If one postulates the possibility of a continuum of atoms, this can be pushed up to fourth-order number theory. Even second-order is already rich enough to represent vast amounts of ordinary mathematics, yet there is no use of set-membership or even an abstract ontology of classes and relations. Structures and structural relations are gotten at indirectly; set-theoretic and higher-order logical constructions can be encoded by talk of pluralities (invoking pairing for relations), but without actually reifying structures or relations. We have a "structuralism without structures".

We have now provided answers to the questions under (1) for MS in some detail. Concerning (2), assertory axioms, the initial Axiom of Infinity (Ax ∞) and the displayed comprehension schemata form the core

[20]See [6].

[21]It is worth noting, for example, that Goodman [15], who helped promulgate mereology (which he called the calculus of individuals), applied it to sense qualia, themselves conceived as "abstract" in the sense of "multiply instantiable" and not themselves spatio-temporally bound.

system, to which can be added more specific modal-existence claims for particular kinds of structures. That for $\mathbb{N}$-structures is already derivable from (Ax ∞) and instances of comprehension, as can even the modal-existence of continua or $\mathbb{R}$-structures, obtained by following well-known classical constructions, e.g. via Dedekind cuts. For *atomic* $\mathbb{R}$-structures, with reals as atoms, however, further postulation is necessary. Similarly further postulates are needed for structures of higher cardinality, e.g. for Zermelo set theory and beyond. It should also be mentioned that the *unicity* (uniqueness up to isomorphism) of $\mathbb{N}$- and $\mathbb{R}$-structures is also derivable in the core system.

As to (3), this version of structuralism is thoroughly eliminativist, as already described.[22] Concerning what distinguishes *mathematical* from other structures, the same second-order logical criterion mentioned in connection with SGS is available (appealing to translation via plurals).

Concerning (4), MS is uniquely explicit in distinguishing *mathematical existence* from ordinary existence. Zermelo spoke of the former as "ideal existence", an idea similar to the notion of logical possibility. By bringing this to the fore, however, MS permits explicit principles of Extendability of (pluralities of) structures (which Zermelo [1930] articulated for models of set theory), but without generating any universal classes of structures or structural objects (as arise in SGS and in a straightforward second-order logical formalization of Zermelo (1930) as well), due to the natural limitations set by extensional second-order comprehension. It simply makes no sense to speak of a collection or plurality of all structures or items in structures *that there might be.* (This builds on the more mundane fact that, on an ordinary understanding of "collecting", you cannot actually collect anything which only might have existed.)

As to (5), MS is distinguished from other versions in eliminating the need for reference to mathematical structures, since it is only possibilities of structurally interrelated objects that are entertained, and these are given by general descriptions. Ordinary designators, e.g. numerals, occurring in everyday use can be accounted for in various ways, e.g. as indicating relevant places in structures, as convenient devices in counting, measuring, and computing, as introduced in mathematical reasoning modulo the assumption of (the possibility of) a structure of a given type, e.g. $\mathbb{N}$-structures, following the logical move of existential instantiation, and so forth. But the usual puzzles (and "bad questions") concerning platonistic reference to *abstracta* do not arise.

[22] Of course, the elimination is for pure mathematics, without prejudice to any actual instantiations of structures there may happen to be in the material world.

The real challenge for MS lies in the last question, concerning epistemic access, not in this case to structures themselves, as these are eliminated on this interpretation, but with respect to the possibilities of structures. What sort of evidence can we have for the various modal-existence postulates arising in mathematics, as illustrated above? Of course, we may gain evidence of formal consistency of the associated axiomatic systems (including various strong forms of consistency, e.g. ω-consistency), even if we cannot have a finitistic proof in the central cases of interest. But the second-order machinery of MS is adopted so that *standard* models of theories (e.g. number theory, analysis, set theories) will be describable, and the possibility of these is not guaranteed by formal consistency claims. It seems that we must fall back on indirect evidence pertaining to our successful practice internally and in applications, and, perhaps, the intuitive pictures and ideas we have of various structures as supporting the coherence of our concepts of them. Perhaps that is the best that any version of structuralism can hope for. We will return to this below, when we come to address some specific challenges that have been raised for MS.

Turning briefly to the problems (i) - (v) raised above, it should be clear that none of them affects MS. Set theories are interpreted structurally, and questions about "the real world of sets" do not arise. Multiple structural possibilities are allowed for, including full "power sets", less-than-full, well-founded domains, non-well-founded, etc. Extendability principles are explicitly part of the interpretation of set theory, leading to the "small" large cardinals (inaccessible, hyperinaccessible of all orders, Mahlo, n-Mahlo, etc.). And, as explained, extensional comprehension does not permit recognition of maximal totalities such as that of "all possible structures". Finally, the whole thrust of MS is to avoid postulating special abstract objects, so the puzzles concerning "places" and "purely structural relations" do not arise. The MS route to "abstractness" consists, not in attempting to introduce "featureless objects", but in simply not building into the descriptions of hypothetical structures anything beyond what is of mathematical interest. Benacerraf's puzzle is solved by accepting his conclusion: numbers as objects are officially eliminated, although number-words can be introduced as aids in computation and reasoning. Finally, regarding (vi), MS is as explicit as any version regarding its assumptions.

The main new problem for MS is reliance on primitive modality (call this (vii)), analogous to SGS's reliance on the primitive 'coherent'. One would like a formal criterion for these notions, but that is not to be hoped for, and the approach confronts the epistemological questions broached above.

Space permits us to consider here briefly only the core Axiom of Infinity (Ax ∞), which, with the MS machinery as already explained, suffices for the vast bulk of ordinary mathematics.

The possibility of a countable infinity of objects seems so entrenched in, and indispensable to, our scientific and mathematical thinking that it is difficult to argue against the skeptic. Intuitionists who insist on human mental constructions of course cannot be satisfied as, of course, we have only finite resources to work with and can only work so fast. From the classical MS perspective, the issue of supertasks is entirely beside the point. As Hale recognized [16], it is the ready conceivability of situations in which infinitely many mind-independent objects exist that we naturally appeal to if pressed. Of course, the platonist can claim, in a variety of ways, that our present situation is such, since, e.g., a proposition p exists and, for every proposition q, the statement *"It is true that q"* expresses a proposition, distinct from q (Bolzano); or that some object o is given, and that for any object a, the singleton of a exists and is distinct from a (Zermelo), etc. But nominalists are not deprived of the possibility of infinities just for not going along with propositions, sets, etc. It is sufficient if, for example, it could be the case that there is a moment of time and that for every moment of time there is a later one (perhaps forming a convergent series), or—as Dummett concedes to be perfectly intelligible—that there be stars such that, for any one of them, another one exists some further distance away, etc., etc. Now, in examining the move from conceivability to possibility, Hale [16] explicitly distinguishes between requiring that the conceived situation be one *in which* it *could* be verified that there are infinitely many things—a condition he (rightly) regards as too strongly verificationist—and requiring rather that the conceived situation be one *of which* it *can* be recognized that, *were it to obtain,* there would indeed be an infinity of things. The latter can be satisfied if a sufficiently detailed description is available from which it can (here, in *our* world) be inferred that an infinity exists wherever the description holds (even this is not in general quite necessary, according to Hale), whereas the former might be quite impossible without supertasks. And it is the latter requirement that is to govern, according to Hale.([16], 137-8.) So what, then, is wrong with the appeal to moments of time, stars, etc., as above? Our descriptions straightforwardly entail the existence of infinitely many things in those situations, and Hale seems to grant the possibility that those descriptions could hold. We certainly *can* see the entailment of infinity, right here and now. It appears that, several pages later, however, there is a slide in Hale's discussion back to a verificationist requirement, for he writes that "the imagined situation would have to be given by a description of which we *could* tell...that, were it to be satisfied, this *would* mandate acceptance of a theory which entails the existence of a completed concrete ω-sequence" ([16], 145, my emphasis), and just prior to this he writes that only *ordinary* tasks are relevant in assessing an imagined situation "as being one of which we *could* recognize that, were it to obtain, a concrete ω-sequence would exist" (*ibid.,* my emphasis).[23] Then, not surprisingly, the MS appeal to situations in which there is always a later moment of time or another star further away, etc., will be found wanting. It seems that, after all, we would have to be able to *determine* such things *in the imagined situations*, in a strong sense, e.g. have evidence that could not be explained on a strictly finitistic basis. It seems that the red herring of supertasks is again out of the jar

A further criticism sometimes levelled against MS concerns its use of second-order logic. There are two interrelated parts to this: first, second-order logic, in its intended sense, is not formalizable; and, second, this reflects the fact that a substantial amount of mathematics is thus presupposed, raising the spectre of circularity. In reply, the first point is correct and is a corollary of Gödel's incompleteness theorems (in the context of Dedekind's categoricity of the Dedekind-Peano axioms). It is the price paid for the gain in expressive power along with the failure of logical compactness. *In seeking a systematic formulation of structuralism, however, one is not attempting to formalize all of mathematics.* The advantages of explicitness and clarity concerning one's assumptions speak for themselves, despite the unattainability of completeness. Indeed, the open-endedness and extendability of mathematics are reasons enough to forego the latter aim. Moreover, concerning the second part of the criticisim, there is no need to insist on an absolute distinction between logic and mathematics, for MS *does not seek a reduction of mathematics to logic* in anything like the traditional sense(s) (e.g. to demonstrate the "analtyicity" of mathematics). It should be granted that some core mathematical content must be built into ones primitive notions if structuralism is to be articulated at all. Indeed the notions

[23] Lest it be thought that we are resolving some subtle ambiguity of modal usage in a biased way, we note that, in the ensuing discussion, Hale writes:

"[the structuralist] must supply a description—necessarily finite—of a possible situation, no empirically adequate theoretical account of which could avoid postulating the existence of a completed concrete ω-sequence. But any desription which could—in the present context—be reckoned unproblematic will perforce mention only finitely many observationally ascertainable facts...which an empirically adequate theory must explain." (p. 145.)

This then rules out closure conditions involving quantification, e.g. "for any moment of time, there is a later one", on the grounds that its satisfaction is not "observationally ascertainable". The slide back to a verificationist requirement is complete.

of "arbitrary plurality" of infinitely many objects and "arbitrary part" of an infinitude of atoms are inherently mathematical, as the work they can do in the detailed development of MS makes clear. Nevertheless, it should be emphasized that no *primitive* notion of 'relation' or 'function' is needed: as noted above, *monadic* plural quantification, combined with mereology, enables a reduction of *polyadic* second-order quantification, i.e. of a full theory of relations. This is a non-trivial gain *vis à vis* versions of structuralism which presuppose 'set-membership' or 'function' or 'relation'. The claim would be that, while far-reaching in their mathematical import, the notion "some of these things", in the plural sense, and that of "part of a whole" of pairwise discrete things, are accessible to us in ordinary contexts and not special to mathematics. Moreover, surely plural quantifiers belong to *logic* in a general sense, even if mereology's status remains moot. Thus, MS could be said to establish a *partial logicism*, less ambitious than the full program to be sure, but significant nonetheless.

Other criticisms of MS have been offered, especially in connection with applications of mathematics, an issue that space has not permitted us to deal with in this paper.[24] In general, structuralists are well-positioned to treat applications since these are naturally understood in terms of full or partial instantiation of mathematical structures by material systems, or, in some cases, just via mappings between these. As an eliminativist version, MS does require some artful manoeuvering to express relevant relationships between material systems and hypothetical objects that merely *might* form mathematical structures, e.g. $\mathbb{N}$-structures, $\mathbb{R}$-structures, or the various spaces of analysis, etc. But the methods worked out in [17], Ch. 3, together with improvements from [7] I believe essentially solve this problem.

6 Summation

Our comparative investigation thus far can be summarized in the following table, most of which should now be self-explanatory (a check means that the objection applies; the goal is to "draw a blank"):

[24]Resnik [30], pp. 74-75, for example, has argued that MS cannot treat ordinary scientific applications of probability and statistics, because of the need for abstract objects such as "events" (e.g. possible outcomes of experiments) and numbers. While Field-style nominalism may be threatened by his objection, I believe it has no force against MS, which can readily invoke the possibility of rich enough structures to *represent* or *model* applications of probability and statistics. What matters is not the metaphysical category of objects in such a model but rather the (applied) mathematical information they carry, which depends on our stipulations and on structural roles. Numbers in an absolute sense are no more required as values of probability functions than they are for ordinary counting or measuring. (Cf. n. 16, above.)

	STS	SGS	CTS	MS
(i) Sets exceptional	√	–	–	–
(ii) Maximal totalities	√	√	?	–
(iii) Possibility of gross error	√	–	–	–
(iv) "Places" as objects	–	√	–	–
(v) Purely structural relations?	–	√	–	–
(vi) Underspecification	–	–	√	–
(vii) Primitive modality	? (informal)	√	?	√

The reason for the '?' in the last row under STS has to do with various informal appeals to possibility in motivating certain points of the theory, e.g. starting with the null set, explaining power sets (via all possible ways of selecting), and motivating largeness conditions ("...the hierarchy go[es] on as long as possible").[25]

As anticipated, none of the approaches is free of problems. But the paucity of checks in the last two columns encourages us to seek some kind of synthesis of CTS and MS. This is in fact achievable, with the effect of removing the check under CTS (vi) while replacing the '?' under (ii) with a blank, but replacing that under (vii) with a check. Instead of relativizing CT to background universes of sets, one can introduce hypothetical large domains (corresponding to inaccessible cardinalities) merely employing the language of mereology and plurals, as in MS; CT can be carried out relative to such domains, any one of which will also support many topoi, incorporating the relativity that Bell has described as well (resulting in an overall double relativity).[26] At the same time, set theory itself can be developed structurally, relative to such domains. Extendability principles can readily be formulated and adopted applying to these domains, and the same considerations that insure a blank under (ii) for MS carry over. Puzzles about proper classes in set theory and large categories in category theory are handled in parallel fashion, by relativization to large domains (recovering, for set theory, Zermelo's method, and for CT the Grothendieck method of universes). If we are right, use of modal notions is the price we must pay if we are to have a well-specified structuralism which respects the indefinite extendability of universes of discourse for mathematics.

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[25] See, e.g., [26], p. 141. For a fuller discussion of these points, see [19].

[26] For details, see [20].

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