Optimal Sensor and Actuator Selection for Large-Scale Dynamical Systems

Neil K. Dhingra, Mihailo R. Jovanović, and Zhi-Quan Luo

49th Asilomar Conference on Signals, Systems and Computers, 2015
Motivation

- **Objective:** Systematic approach for sensor/actuator selection

- **Applications**
  - Phasor Measurement Units in power networks
  - Autonomous formations of vehicles
  - Flexible wing aircraft
Recent work

Sensor selection for parameter estimation

- Linear measurement model (Joshi, Boyd ‘09)
- Optimal experiment design (Kekatos, Giannakis, Wollenberg ‘12)

Sensor/actuator selection in dynamical systems

- Nonconvex formulation (Masazade, Fardad, Varshney ‘12)
- Adaptive selection (Chepuri, Leus ‘14)
- Maximize effect of actuators (Summers, Lygeros ‘14)
  (Tzoumas, Rahimian, Pappas, Jadbabaie ‘15)

Convex characterization as SDP

- SDP formulation (Polyak, Khlebnikov, Shcherbakov ‘13)
- Discrete time (Munz, Pfister, Wolfrum ‘14)
Sensor selection

Linear time invariant system with many potential sensors

\[
\dot{x} = Ax + d \\
y = Cx + \eta
\]
Sensor selection

Linear time invariant system with many potential sensors

\[
\begin{align*}
\dot{x} &= Ax + d \\
y &= Cx + \eta
\end{align*}
\]

Kalman filter estimates state from measured output

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + L(y - \hat{y}) \\
\hat{y} &= C\hat{x}
\end{align*}
\]
Sensor selection

Linear time invariant system with many potential sensors

\[
\begin{align*}
\dot{x} &= Ax + d \\
y &= Cx + \eta
\end{align*}
\]

Kalman filter estimates state from measured output

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + L(y - \hat{y}) \\
\hat{y} &= C\hat{x}
\end{align*}
\]

Objective: Minimize estimation error using a few sensors
Sensor selection via regularization

\[
\text{minimize} \quad J(L) + \gamma \sum_i \|L e_i\|_2
\]

\[
\downarrow \quad \downarrow
\]

variance of estimation error \quad column-sparsity-promoting penalty function

- Larger \( \gamma \) selects fewer sensors
- \( \gamma = 0 \) uses all sensors

\( \ell_1 \)-penalized optimal control - Lin, Fardad, Jovanović, ACC ‘11, IEEE TAC ‘13
Observer design

\[
\begin{align*}
\dot{x} &= Ax + d \\
y &= Cx + \eta
\end{align*}
\]

system

\[
\begin{align*}
\dot{x} &= \hat{A}\hat{x} + L(y - \hat{y}) \\
\dot{\hat{x}} &= A\hat{x} + L(y - \hat{y})
\end{align*}
\]

observer

Variance of estimation error \( e := (x - \hat{x}) \)

\[
\dot{e} = (A - LC)e + d + L\eta
\]

\[
J(L) = \text{trace} \left[ \lim_{t \to \infty} \mathbb{E} (e(t)e^T(t)) \right]
\]

Kalman filter minimizes \( J(L) \) using all sensors
SDP Characterization

$V_d, V_\eta$ are process disturbance and measurement noise covariance

\[
\begin{align*}
\text{minimize} \quad & \quad \begin{array}{c} \text{trace}(V_d X) + \text{trace}(V_\eta L^T X L) \end{array} \\
\text{subject to} \quad & \quad (A - LC)^T X + X (A - LC) + I = 0 \\
& \quad X \succ 0
\end{align*}
\]
SDP Characterization

\( V_d, V_\eta \) are process disturbance and measurement noise covariance

\[
\begin{align*}
\text{minimize} & \quad J(L) = \text{trace}(V_d X) + \text{trace}(V_\eta L^T X L) \\
\text{subject to} & \quad (A - LC)^T X + X (A - LC) + I = 0 \\
& \quad X \succ 0
\end{align*}
\]

Change of variables \( Z := X L \)

\[
\begin{align*}
\text{minimize} & \quad \text{trace}(V_d X) + \text{trace}(V_\eta Z^T X^{-1} Z) \\
\text{subject to} & \quad A^T X - C^T Z^T + X A - Z C + I = 0 \\
& \quad X \succ 0
\end{align*}
\]

\>

Schur complement yields convex problem
- Challenge: impose structure on $L$ in $(X, Z)$ coordinates
  - Linear constraints on $L$ become \textit{nonlinear} on $(X, Z)$
Challenge: impose structure on $L$ in $(X, Z)$ coordinates

- Linear constraints on $L$ become *nonlinear* on $(X, Z)$

\[
\begin{array}{c}
& Z \\
& = \\
X & L
\end{array}
\]

Column-sparsity is preserved

\[
\begin{array}{c}
& Z \\
& = \\
X & L
\end{array}
\]
Sensor selection - Semidefinite Program

\[
\begin{align*}
\text{minimize} & \quad \text{trace} (V_d X) + \text{trace}(V_\eta Z^T X^{-1} Z) + \gamma \sum \| Z e_i \|_2 \\
\text{subject to} & \quad A^T X - C^T Z^T + X A - Z C + I = 0 \\
& \quad X \succ 0
\end{align*}
\]

- Promote column sparsity of \( Z \) instead of \( L \)

Polyak, Khlebnikov, Shcherbakov, ECC ‘13
Dhingra, Jovanović, Luo, CDC ‘14
Computational complexity

\[ \text{trace}(V_\eta Z^T X^{-1} Z) = \text{trace}(V_\eta \Theta) \]

Worst case computational complexity: \( \mathcal{O} ((n + m)^6) \)
Alternating Direction Method of Multipliers (ADMM)

Splitting method for convex problems with linear constraints

Form augmented Lagrangian

\[ \mathcal{L}_\rho(X, Z, \Lambda) := f(X, Z) + \langle \Lambda, h(X, Z) \rangle + \frac{\rho}{2} \| h(X, Z) \|_F^2 \]

\[ h(X, Z) := A^T X - C^T Z^T + X A - Z C + I \]
Update variables \textit{separately}

\[ X^{k+1} = \arg \min_X \mathcal{L}_\rho(X, Z^k, \Lambda^k) \]

\[ Z^{k+1} = \arg \min_Z \mathcal{L}_\rho(X^{k+1}, Z, \Lambda^k) \]

\[ \Lambda^{k+1} = \Lambda^k + \rho h(X^{k+1}, Z^{k+1}) \]

Boyd, Parikh, Chu, Peleato, Eckstein ‘11
**Z-minimization**

\[
\text{minimize } \gamma \sum_{Z} \|Z e_i\|_2 + \frac{\rho}{2} \left\| Z C + C^T Z^T + W^k \right\|_F^2
\]

\( J_Z(Z) \)

- Group LASSO
**Z-minimization**

\[
\text{minimize} \quad \gamma \sum_{Z} \|Z e_i\|_2 + \frac{\rho}{2} \|Z C + C^T Z^T + \|W^k\|_F^2 \]

\[
J_Z(Z)
\]

- **Group LASSO**

- **Proximal method:** iterative soft thresholding algorithm (ISTA)

\[
Z^{m+1} = S_{\alpha m \gamma / \rho} (Z^m - \alpha m \nabla J_Z(Z^m))
\]

- **Computational complexity** \(O(n^2 m)\)
\textbf{X-minimization}

\[
\min_{X} \text{trace} \left( X V_d + X^{-1} Z^k V_\eta (Z^k)^T \right) + \frac{\rho}{2} \| A^T X + X A + U_k \|_F^2
\]

- Can formulate as SDP (worst case $\mathcal{O}(n^6)$)
  - Many sensors, $m \gg n \implies n^6 \ll (n + m)^6$
  - Model reduction can reduce $n$
**$X$-minimization**

\[
\minimize_X \text{trace} \left( X V_d + X^{-1} Z^k V_\eta (Z^k)^T \right) + \frac{\rho}{2} \| A^T X + X A + U_k \|_F^2
\]

- Can formulate as SDP (worst case $\mathcal{O}(n^6)$)
  - Many sensors, $m \gg n \implies n^6 \ll (n + m)^6$
  - Model reduction can reduce $n$

- Projected Newton’s method
  - Conjugate gradient (worst case $\mathcal{O}(n^5)$)
  - Inexact minimization; faster in practice
  - Project onto $\{X : X \succ 0\}$
Extensions - Actuator selection

State feedback $u = -Kx$

$$\dot{x} = Ax + Bu + d$$

$$J(K) = \int_{0}^{\infty} x^T Q x + u^T R u \, dt$$

- Promote row sparsity of $Y := KX$
State feedback $u = -Kx$

$$
\begin{align*}
\dot{x} &= Ax + Bu + d \\
J(K) &= \int_0^\infty x^TQx + u^TRu \, dt
\end{align*}
$$

▸ Promote row sparsity of $Y := KX$

minimize $\underset{X,Y}{\text{trace}(QX)} + \text{trace}(RYX^{-1}Y^T) + \gamma \sum \|e_i^TY\|_2$

subject to $AX - BY + XA^T - Y^TB^T + V_d = 0$

$X \succ 0$
Extensions: Sensor selection for feedback control

- Given linear quadratic regulator $u = -K \hat{x}$
- Select sensors to optimize closed loop performance
Extensions: Sensor selection for feedback control

Given linear quadratic regulator $u = -K\hat{x}$

Select sensors to optimize closed loop performance

$$\begin{align*}
\text{minimize}_{X,Z} & \quad \text{trace}(V_d X) + \text{trace}(V_\eta Z^T X^{-1} Z) + \gamma \sum \|Ze_i\|_2 \\
\text{subject to} & \quad A^T X - C^T Z + X A - Z C + K^T R K = 0 \\
& \quad X \succ 0
\end{align*}$$
Example - Flexible Wing Aircraft

- Detect aeroelastic instability
Using less than half the sensors degrades performance by only $\sim 20\%$
Example - Mass Spring System

\[ \ddot{x}_i = (x_{i+1} - x_i) + (x_{i-1} - x_i) + d_i \]

▶ Dynamics

▶ Sensors

▶ Position of each mass

▶ Velocity of each mass
Computation time for $\gamma = 100$

- $n$ states and $n$ outputs
- 50 states and $m$ outputs
Conclusions

- Convex characterization of sensor or actuator selection
- Splitting algorithm
  - Solve simpler sub-problems
  - Scaling dependent on number of states

Future work

- More efficient $X$-minimization
- Alternative splitting methods
- Joint sensor and actuator selection
Acknowledgements

Support:

▶ NASA JPFP Fellowship
▶ MnDRIVE Graduate Scholars Program
▶ NSF ECCS-1407958

Collaboration on aircraft example

▶ Marty Brenner, NASA
▶ Claudia Moreno and Harald Pfifer
Dual Problem

\[ g(Y) = \text{maximize} \ \langle U, Y \rangle \]
\[ e_i^T U \leq \gamma \]

Lagrangian

\[ L(X, Y, \Lambda, M) = \text{trace}(QX + X^{-1}Y^TRY) + \langle U, Y \rangle + \langle \Lambda, AX + XA^T - BY - Y^TB^T + V \rangle - \langle M, X \rangle \]

where \( M \succeq 0 \)

\[
\begin{bmatrix}
\nabla_X L \\
\nabla_Y L
\end{bmatrix} = \begin{bmatrix}
Q - X^{-1}Y^TRYX^{-1} + A^T\Lambda + \Lambda A \\
2RYX^{-1} + U - 2B_2^T\Lambda
\end{bmatrix}
\]

Dual Problem

maximize \( \langle V, \Lambda \rangle \)
subject to \( Q + \Lambda A + A^T\Lambda - (B_2^T\Lambda - \frac{1}{2}U)^T R^{-1} (B_2^T\Lambda - \frac{1}{2}U) \succeq 0 \)
\[ \|e_i^T U\|_2 \leq \gamma \]