Problems From MthE 5011
University of Minnesota

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The following problems were assigned to me in a class. I had the option of choosing twelve to solve, which explains the non-consecutive numbering. The problems are restated with their solutions in the rest of the document.

Problem 1 — Log Cutting

A tree cutter can saw a log into four pieces in twelve seconds. Working at the same rate, how long would it take to cut the log into five pieces?

Problem 2 — Checkers

How many squares are there altogether on an 8x8 checkerboard? How about an n X n checkerboard?

Problem 3 — Sequence

Find the one millionth term of the following sequence:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, …

Problem 6 — Weights

A merchant had a 40 kg weight that he used for weighing objects on a double pan balance. He dropped the weight and broke it into four pieces. Upon weighing the pieces, he discovered each piece to be an integral number of kilograms, and with them he could weigh all integral numbers of kilograms from one through forty. Find the weights of the four pieces.

Problem 9 — Train

A man is 3/8 of the way across a bridge when he sees a train approaching at 60 mph. If he runs as fast as he can in either direction, he will just make it to the end of the bridge before the train does. How fast can he run?

Problem 10 — Fitness Fanatics

Fred and Frank are two fitness fanatics on a run from A to B. Fred runs half of the way and walks the other half. Frank runs for half of the time and walks the other half. They both run and walk at the same speeds. Who finishes first?

Problem 12 — Equator

Put a ring around the equator. Now cut it, enlarge it by 6 feet, and put it back. How far from the earth will it be? (the earth's radius is 4000 miles).

a) a negligible amount.
b) a ladybug might be able to crawl underneath the ring.
c) a chipmunk could run underneath the ring.
d) your mathematics teacher could crawl underneath the ring.
e) you could drive a riding lawn mower underneath the ring.

Problem 16 Marbles

You have ten boxes and forty-four marbles. Explain how you can put all of the marbles in the boxes so that each box contains a different number of marbles.

Problem 17 Darts

If a dart has an equal chance of landing at any point on a circular target, is it more likely to land closer to the center or closer to the edge?

Problem 18 Sum

Find the sum of the following sequence: $\sum_{i=2}^{\infty} \frac{1}{i}$

Problem 20: Pizzas

Twenty unique pizzas are randomly delivered to twenty people who ordered them. What is the probability that exactly nineteen people will receive the correct pizza?

Problem 21: Number Cubes

Design a numbering system for two cubes so that when rolled, the sum can be any whole number from 1 to 12 and each sum has the same probability of occurring.

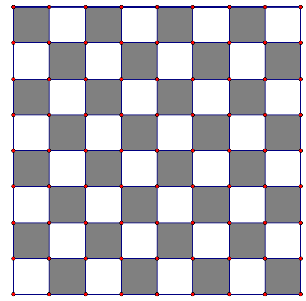
Question:
A tree cutter can saw a log into four pieces in twelve seconds. Working at the same rate, how long would it take to cut the log into five pieces?

Solution:
The tree cutter works at a rate of three cuts in twelve seconds, or one cut every four seconds. To cut a log into five pieces requires four cuts, which means it would take this wood cutter 16 seconds (four cuts at a rate of four seconds per cut).

Rate: $\frac{3 cuts}{12 \sec} = \frac{1 cut}{4 \sec}$

How long for five pieces? Work X Rate $\Rightarrow (4 cuts) * \left(\frac{4 \sec}{1 cut}\right) \Rightarrow$ **16 seconds**

Possible Pitfalls:
A common mistake in this problem is to contrive a rate which involves number of pieces. This leads to the misconception that cutting a log into five pieces is 5/4 as much work as cutting a log into four pieces. However, the work is not measured in pieces, but in cuts. Looking at the number of cuts, we find that the second job is actually 4/3 as much work.

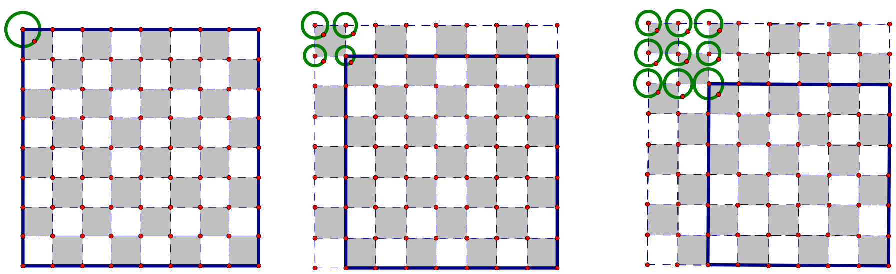


Question:
How many squares are there altogether on the checkerboard? How about an n X n checkerboard?

Solution:
We will solve the general case first, and use that to find the 8 X 8 case. On an n X n checkerboard, the squares will have any side length from one, up to and including n. This means that the number of squares on the board is:

# of nXn squares + # of (n-1)X(n-1) squares + # of (n-2)X(n-2) squares +…+ # of 1X1 squares.

Now we must determine how many squares of each size there can be. This is simple if we consider only one piece of the square, say the top left corner. The top left corner of each square (of any size) coincides with the top left corner of one of the small squares. The largest square will fill the checkerboard with no room to shift, so there can be only one. The second largest can shift one square to the right, one down, or one to the right and one down. Consider the diagram below; the circled points are possible locations for the top left corner of each square of a certain size.



We can see clearly that there can be:

1 n X n square
4 (n-1) X (n-1) squares
9 (n-2) X (n-2) squares
…
n^2 1 X 1 squares.

Therefore, the total number of squares on an n X n checkerboard is equal to

$$\mathbf{1+4+9+16+25+36+\ldots+n^2} = \sum_{i=1}^{n} i^2 \ .$$

For the board shown, n = 8, so $\sum_{i=1}^{n} i^2$ becomes 1+4+9+16+25+36+49+64= **204 squares**

Question:
Find the one millionth term of the following sequence:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, …

Solution:
Observing the structure of this sequence, we find that the final instance of every integer n occupies the $(n+(n-1)+(n-2)+\ldots+3+2+1)^{th}$ place in the sequence. Given the number, you can find the position in the sequence. However, we do not have the number, we have the position in the sequence.

We will employ a formula for the sum of positive integers, used by the legendary Karl Gauss at the age of ten.

$$\sum_{i=1}^{n} i = 1+2+3+4+\ldots+n = \frac{n(n+1)}{2}.$$

The idea behind this formula is that it adds the sequence to itself.
The *last* term is added to the *first*,
The *second-to-last* term is added to the *second*,
The *third-to-last term* is added to the *third*,
And so on, until
The *second* term is added to the *second-to-last*,
And the *first* term is added to the *last* term, as shown below:

$$\begin{array}{l} \quad 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \ldots + (n-3) + (n-2) + (n-1) + n \\ \underline{+\ n + (n-1) + (n-2) + (n-3) + (n-4) + (n-5) + (n-6) + (n-7) + \ldots + 4 + 3 + 2 + 1} \\ (n+1)+(n+1)+(n+1)+(n+1)+(n+1)+\ldots\ldots\ldots+(n+1)+(n+1)+(n+1)+(n+1) \end{array}$$

Now we must determine how many (n+1)'s there are. Since there are n terms in the sequence and each term adds with something else to produce an (n+1), we know that there are exactly n of these (n+1)'s. Since we are trying to find the sum of the terms of the sequence, we must divide by two, because n(n+1) is the sum of the sequence added to itself. Hence,

$$\sum_{i=1}^{n} i = 1+2+3+4+\ldots+n = \frac{n(n+1)}{2}.$$

[continued on next page→]

Using this result, we can find out how many terms are in our sequence up to and including the last occurrence of any integer n. It seems that we are still faced with trial and error, however. We can come very close to our answer by setting this sum equal to 1,000,000.

$$1{,}000{,}000 = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n. \qquad \text{Then} \qquad 0 = \frac{n^2}{2} + \frac{n}{2} - 1{,}000{,}000.$$

We solve using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{gives solutions of} \quad 0 = ax^2 + bx + c$$

This gives us:

$$n = \frac{\left(-\frac{1}{2}\right) \pm \sqrt{\left(\frac{1}{2}\right)^2 - 4 \cdot \frac{1}{2} \cdot (-1{,}000{,}000)}}{2 \cdot \left(\frac{1}{2}\right)}$$
$$= \left(-\frac{1}{2}\right) \pm \sqrt{2{,}000{,}000.25} = 1413.71...$$

Notice here that we are only concerned with the positive solution, since it is not reasonable to believe that the 1,000,000th term in a sequence of increasing positive integers will be negative. Because of the .71… to the right of the decimal point, it seems reasonable to think that our 1,000,000th term is greater than 1413 and not more than 1414. In the context of our sequence, this means that it is beyond the last 1413 and not yet to the last 1414, so it should be a **1414**. We can check our answer using our formula from earlier,

$$\frac{n(n+1)}{2} = \frac{1413(1413+1)}{2} = 998{,}991$$

This means that the last 1413 appears as the 998,991st term in the sequence

$$\frac{n(n+1)}{2} = \frac{1414(1414+1)}{2} = 1{,}000{,}405$$

So the 1414s go from the 998,992nd position through the 1,000,405th position.
Therefore, the 1,000,000th term in the sequence is 1414.

Question:
A merchant had a 40 kg weight that he used for weighing objects on a double pan balance. He dropped the weight and broke it into four pieces. Upon weighing the pieces, he discovered each piece to be an integral number of kilograms, and with them he could weigh all integral numbers of kilograms from one through forty. Find the weights of the four pieces.

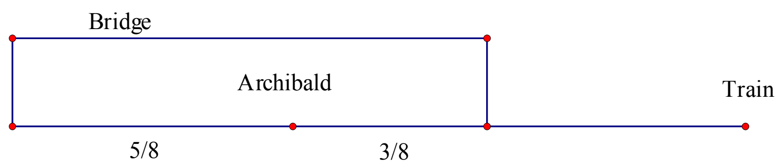
Solution:
It is important to realize here that a double pan balance lets you place weight on both sides, so you can either add or subtract with the weights you have. Since he is able to weigh all weights from one through 40, we know that he must have a 1 kg piece. Without a 1 kg weight, he would be unable to measure 39 kg. **Through trial and error, I discovered that 1, 3, 9, 27 work**. This combination occurred to me because of the use of powers of two in computer architecture. In computers, data is represented by binary numbers, so that four digits offers any combination of either zero or one 1's, zero or one 2's, zero or one 4's, and zero or one 8's.

Note that $3^0, 3^1, 3^2, 3^3 = 1, 3, 9, 27$.

Our situation offers an even greater range of use, since we can have either minus one, zero, or one 1, minus one, zero, or one 3, minus one, zero, or one 9, and minus one, zero, or one 27. To be honest, I am not sure if it is the same principle at work.

1	1kg	21	3kg+27kg-9kg
2	3kg-1kg	22	1kg+3kg+27kg-9kg
3	3kg	23	27kg-1kg-3kg
4	1kg+3kg	24	27kg-3kg
5	9kg-1kg-3kg	25	1kg+27kg-3kg
6	9kg-1kg	26	27kg-1kg
7	1kg+9kg-3kg	27	27kg
8	9kg-1kg	28	1kg+27kg
9	9kg	29	3kg+27kg-1kg
10	1kg+9kg	30	3kg+27kg
11	3kg+9kg-1kg	31	1kg+3kg+27kg
12	3kg+9kg	32	9kg+27kg-1kg-3kg
13	1kg+3kg+9kg	33	9kg+27kg-3kg
14	27kg-1kg-3kg-9kg	34	1kg+9kg+27kg-3kg
15	27kg-3kg-9kg	35	9kg+27kg-1kg
16	1kg+27kg-3kg-9kg	36	9kg+27kg
17	27kg-1kg-9kg	37	1kg+9kg+27kg
18	27kg-9kg	38	3kg+9kg+27kg-1kg
19	1kg+27kg-9kg	39	3kg+9kg+27kg
20	3kg+27kg-1kg-9kg	40	3kg+9kg+27kg+1kg

Question:
A man is 3/8 of the way across a bridge when he sees a train approaching at 60 mph. If he runs as fast as he can in either direction, he will just make it to the end of the bridge before the train does. How fast can he run?



Solution:
This type of problem invites confusion, but the solution is rather eloquent and satisfying. It is tempting to jump straight into a rate equation, but the quantities involved seem to defy description. We would need some variable for the length of the bridge, the unknown distance from the train to the bridge, etc. However, it can be easily explained.

The man, whom we will call Archibald, can run 3/8 of the bridge in the time it takes the train to reach the bridge. If he is running toward the near side of the bridge (and the direction from which the train is approaching), this will allow him to reach the end of the bridge. On the other hand, if Archibald is decides to run toward the far side of the bridge (and away from the train), he will still be on the bridge when the train reaches the bridge. Running to the far side, he begins 5/8 of the bridge from safety. After running 3/8 of the bridge, a revealing moment occurs: the train, coming from behind him, has reached the bridge, and Archibald has the remaining 2/8, (or ¼) of the bridge left to run. Since we know he makes it to safety, we can conclude that he runs that ¼ bridge in the same time it takes the train to cross the entire bridge.

distance = rate*time

$$\frac{d_{Archibald}}{r_{Archibald}} = \frac{d_{Train}}{r_{Train}} = t \rightarrow \frac{\frac{1}{4}bridge}{r_{Archibald}} = \frac{bridge}{60}$$

It follows that Archibald is running at a speed equal to ¼ the speed of the train, which we know to be 60 mph. **Therefore, Archibald can run 15 mph.**

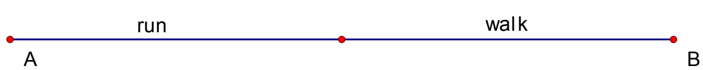
Question:
Fred and Frank are two fitness fanatics on a run from A to B. Fred runs half of the way and walks the other half. Frank runs for half of the time and walks the other half. They both run and walk at the same speeds. Who finishes first?

Solution:

While we don't know their rate of walking or their rate of running or how much faster running is than walking, we do assume that running is faster than walking. This question asks us to sort out the rates, times, and distances in the situation described. As it stands, we are asked to compare a distance with a time. We cannot do that, so we need to describe either Frank or Fred in terms of the other, so that they might be compared. Let us consider distances. Fred runs ½ the distance. Frank, who runs ½ the time, must surely cover more distance in the running half of his time than in the walking half. Therefore, Frank runs for more than ½ the distance. This we can compare.

At the beginning of the race, Frank and Fred are both running at the same pace. At the halfway point (distance-wise), Fred will slow down to a walk, but Frank will keep running (and pull ahead of Fred). Frank will, at some point, also slow down to a walk, but only after he is in front of Fred. From the point Frank starts walking, they will maintain the same distance between them, and **Frank will win the race.**

Fred runs 1/2 the distance from A to B, then walks the rest of the way.



Frank runs 1/2 the time from A to B, then walks the rest of the way.

Another Solution:

This can also be proven rather beautifully using a proof by contradiction. Suppose Fred wins. Then Frank spends more time than Fred does in traveling from A to B, which means he runs for more than ½ of Fred's time. But halfway through Fred's time, Fred is already walking. Since Frank is running at that moment, he pulls ahead of Fred and continues to run for some unknown length of time. At some point, he will slow to a walk, but the damage is done, he has passed Fred, whose walking speed is not fast enough to catch up. Here we are faced with a contradiction, since Fred was supposed to win! Therefore, Fred cannot win, Frank must win.

Question:
Put a ring around the equator. Now cut it, enlarge it by 6 feet, and put it back. How far from the earth will it be? (the earth's radius is 4000 miles).
a) a negligible amount.
b) a ladybug might be able to crawl underneath the ring.
c) a chipmunk could run underneath the ring.
d) your mathematics teacher could crawl underneath the ring.
e) you could drive a riding lawn mower underneath the ring.

Solution:
We are asked to alter the circumference of a circle. The original length of the equator is the circumference of a circle with radius 4000 miles.

$$C = 2\pi \cdot r \Rightarrow 2\pi \cdot 4000 = 8000\pi \approx 25132.74\,ft$$

…we then enlarge it by 6 ft:

$$C' \approx 25132.74 + 6 = 25138.74\,ft$$

We now want to find the radius of a circle with circumference C'.

$$C = 2\pi \cdot r \Rightarrow \frac{C'}{2\pi} = r'. \Rightarrow \frac{25138.74}{2\pi} = 4000.95\,ft = r'.$$

Therefore, the enlarged circle will have a radius .95 feet longer than the original radius. This means that **c is true, a chipmunk could run underneath the ring,** but a math teacher would probably get stuck.

Question:
You have ten boxes and forty-four marbles. Explain how you can put all of the marbles in the boxes so that each box contains a different number of marbles.

Observations:
My initial approach was to find ten different numbers of marbles, and to place each number of marbles in a different box. However, my observations revealed that there are no ten unique numbers of marbles that add up to forty-four. We know this because the ten smallest unique integers have a sum greater than 44:

$1+2+3+4+5+6+7+8+9+10 = 55 > 44$.

Therefore, we must approach the problem differently.

Solution:
Place 1 marble in box one, 1 marble in box two, 1 in box three, 1 in box four, 1 in box five, 1 in box six, 3 in box seven, 5 in box eight, 7 in box nine, and 23 in box ten.

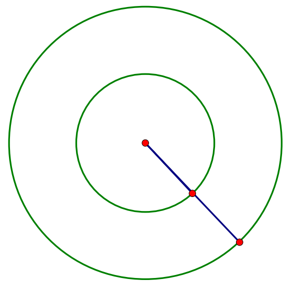
$1+1+1+1+1+1+3+5+7+23 = 44$

Now place boxes six through ten inside boxes one through five, so that the sixth box sits inside the first box, the seventh inside the second, etc. If the boxes will not fit inside, balance them on top so that one bottom corner of the top box sticks down into the bottom box. As you can see, no two boxes contain the same number of marbles.

Now we have:

Box 1: 2 marbles	Box 6: 1 marble
Box 2: 4 marbles	Box 7: 3 marbles
Box 3: 6 marbles	Box 8: 5 marbles
Box 4: 8 marbles	Box 9: 7 marbles
Box 5: 24 marbles	Box 10: 23 marbles.

Question:
If a dart has an equal chance of landing at any point on a circular target, is it more likely to land closer to the center or closer to the edge?



Solution:
The points equidistant from the center and the edge form a circle around the center with radius ½ the dartboard radius. Let's call the radius of the dartboard r. then the halfway circle has radius r/2. Darts which land inside the 'halfway circle' are closer to the center. Darts which land outside are closer to the edge.

The area of the entire dartboard is $\pi \cdot r^2$.

The area of the small circle is $\pi \cdot \left(\frac{r}{2}\right)^2 = \frac{\pi \cdot r^2}{4}$.

We can see that the only one-fourth of the dartboard is closer to the center than the edge, while the remaining three-fourths is closer to the edge than the center. **Thus, it is three times more likely that a dart will be closer to the edge.**

Question:

Find the sum of the following sequence: $\sum_{i=2}^{\infty} \frac{1}{i}$

Solution:
Let us consider this grouping of the first few terms:

$$\sum_{i=2}^{\infty} \frac{1}{i} = \left[\frac{1}{2}\right] + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] + \left[\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right] + \ldots$$

Notice that each group of terms inside the brackets represents a quantity that is greater than or equal to ½. The fourth group above will be followed by 16 terms, each greater than 1/32. Those terms, added together, will be greater than 16/32 or ½.

More generally, beginning with ½, every term $\frac{1}{2^n}$ can be grouped with the preceding $2^{n-1} - 1$ terms, which are each greater than $\frac{1}{2^n}$.

This gives us 2^{n-1} terms, whose sum is greater than $2^{n-1} \cdot \frac{1}{2^n} = \frac{1}{2}$.

Since we have an infinite number of these groups, whose sums are $\geq \frac{1}{2}$, we therefore have an infinite sum. **So** $\sum_{i=2}^{\infty} \frac{1}{i}$ **is infinite.**

Question:
Twenty unique pizzas are randomly delivered to twenty people who ordered them. What is the probability that exactly nineteen people will receive the correct pizza?

Solution:
Given that every pizza is unique, nineteen people receiving the correct pizza would indicate that the leftover pizza did not belong to any of them. Therefore, the leftover pizza must have been ordered by the twentieth person. Each customer has a pizza except for the twentieth person, so that person will receive the leftover pizza, which is the correct pizza. In this way, if nineteen people receive the correct pizza, then all twenty necessarily receive it, and it will never happen such that exactly nineteen receive the correct one. **Therefore, the probability that exactly nineteen people will receive the correct pizza is zero.**

Question:
Design a numbering system for two cubes so that when rolled, the sum can be any whole number from 1 to 12 and each sum has the same probability of occurring.

Solution:
This solution is different from a standard pair of two dice for two main reasons. First, the standard pair offers a range of only 2-12, instead of 1-12. The second difficulty comes with the equal probabilities. A standard pair of dice will produce the outcome two (1,1) only once per 36 rolls. Meanwhile, a seven occurs (3,4),(4,3),(2,5),(5,2),(1,6),(6,1), so six times in 36 rolls.

The cubes described have six sides each, for a total of 36 possible rolls. Since we want our twelve outcomes to each have the same probability, that probability must be 3/36. This means we must find a labeling scheme that will allow each sum to occur in exactly three different ways.

These labeling systems all work:

Cube 1: (1,1,1,7,7,7)
Cube 2: (0,1,2,3,4,5)

Cube 1: (0,0,0,6,6,6)
Cube 2: (1,2,3,4,5,6)

Cube 1: (0.5, 0.5, 0.5, 6.5, 6.5, 6.5)
Cube 2: (0.5, 1.5, 2.5, 3.5, 4.5, 5.5)

Cube 1: (-0.3, -0.3, -0.3, 5.7, 5.7, 5.7)
Cube 2: (1.3, 2.3, 3.3, 4.3, 5.3, 6.3)

Or, more generally, the cubes could be numbered:
Cube 1: {(-n), (-n), (-n), (6-n), (6-n), (6-n)}
Cube 2: {(1+n), (2+n), (3+n), (4+n), (5+n), (6+n)} for any real number n.